Lecture Notes on the Standard Model of Elementary Particle Physics

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Foreword

These lecture notes have been devised as background material for a one-semester master course[1] in theoretical physics, firstly given at the University of Bern during the fall semester 2010. The typical audience includes physics students holding a bachelor and then familiar with nonrelativistic quantum mechanics, special relativity, classical electrodynamics or, more generally, the classical theory of fields. Even if knowledge of quantum field theory is not a prerequisite, it should be at least studied in parallel.

The lecture notes have four parts. The first part describes the four main tools used in gauge theories of elementary particle interactions and in particular in the Standard Model of Glashow, Salam and Weinberg: fields and gauge symmetries, Lie groups and Lie algebras, gauge-invariant Lagrangian field theories and the spontaneous breaking of continuous symmetries. The notes include a discussion of spinor fields since relativistic quantum mechanics is not always known to bachelor physicists. The second part is the construction of the Standard Model and the derivation of its simplest physical properties and predictions (masses, couplings and parameters, structure of interactions, neutrinos, ...). In these two parts, the existence of a quantum field theory only appears through few elements (relation field–particle, conditions for a perturbatively renormalizable, anomaly-free Lagrangian) which can be, at this stage, simply admitted.

The third part discusses more sophisticated aspects of Standard Model phenomenology. Here, results on decay and scattering processes are used, which can only be derived using quantum field theory methods. But the mere analysis of these results, as attempted in these notes, has its own interest. The final fourth part briefly presents some theoretical extensions of the Standard Model, keeping of course the discussion at the level of four-dimensional extensions in the framework of quantum field theory (excluding then unification with gravitation, extra dimensions, strings, ...).

[1]Two periods of fourty-five minutes and one period of exercices each week, fourteen weeks in the semester.

Conventions, units and notations

Units

We follow the practice common in particle physics to work in units in which the speed of light $c = 1$. Time is then measured in length units and momentum and energy units coincide. We also use $\hbar = 1$, which implies that a momentum is an inverse length. Then, a unit of energy also measures distance^{-1}, time^{-1} and momentum.

Conventions, in general

Whenever (except otherwise mentioned) indices are repeated in a formula, a sum over all possible values of these indices is understood. All equations should then have the same "open" indices on both sides.

For two operators A and B, $[A, B] = AB - BA$ (commutator) and $\{A, B\} = AB + BA$ (anticommutator).

Pauli matrices: $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\, I_2$, $[\sigma_i, \sigma_j] = 2i\, \epsilon_{ijk}\, \sigma_k$, ${\sigma_i}^\dagger = \sigma_i$,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_1\sigma_2\sigma_3 = i\, I_2.$$

Symmetrization of indices:

$$A_{(a_1 a_2 \ldots a_k)} = \frac{1}{n!}\Big(A_{a_1 a_2 \ldots a_k} + \text{all permutations of indices } a_1, a_2, \ldots, a_k\Big).$$

Antisymmetrization of indices:

$$A_{[a_1 a_2 \ldots a_k]} = \frac{1}{n!}\Big(A_{a_1 a_2 \ldots a_k} + (-1)^{\delta_P} \times \text{all permutations } P \text{ of indices } a_1, a_2, \ldots, a_k\Big),$$

where δ_P is the signature of the permutation P.

Conventions for space-time

The Minkowski metric is

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

in (cartesian) coordinates $x^\mu = (x^0, \vec{x})$, $x^0 = ct = t$. The inverse $\eta^{\mu\nu}$ is (numerically) the same. Then, since $x_\mu = (x^0, -\vec{x})$,

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right), \qquad \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right),$$

where $\vec{\nabla}$ is the gradient operator.

When explicit indices are not needed, the four-vector x^μ (which is not x_μ) is denoted simply by x. Except otherwise mentioned, repeated indices are summed over all their possible values.

A Lorentz-invariant sum is a contraction of indices. One index is covariant (lower index), one index is contravariant (upper index). For instance, the Lorentz invariant product of two 4-vectors x and y is

$$xy = x^\mu y_\mu = x_\mu y^\mu = \eta^{\mu\nu} x_\mu y_\nu = \eta_{\mu\nu} x^\mu y^\nu.$$

As hereabove, for two 4-vectors x and y, the Lorentz-invariant product is often simply denoted by xy.

Part I

Gauge symmetry and gauge theories

Chapter 1

Gauge invariance and Maxwell electrodynamics

1.1 Gauge invariance of Maxwell equations

Maxwell vacuum equations are

$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0, & \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0, \\ \vec{\nabla} \cdot \vec{E} &= \frac{1}{\epsilon_0}\,\rho, & \vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} &= \frac{1}{\epsilon_0 c^2}\,\vec{j}. \end{aligned} \tag{1.1}$$

The equations in the second line relate the electric and magnetic fields $\vec{E}(t,\vec{x})$ and $\vec{B}(t,\vec{x})$ to their sources, the charge density $\rho(t,\vec{x})$ and the current density $\vec{j}(t,\vec{x})$. They imply the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0, \tag{1.2}$$

which is the local version of (total) electric charge conservation: Maxwell equations only make sense for sources verifying the continuity equation. The first line specifies intrinsic properties of $\vec{E}$ and $\vec{B}$. These equations can be solved in terms of an electric potential $\Phi(t,\vec{x})$ and a magnetic (or vector) potential $\vec{A}(t,\vec{x})$:

$$\vec{B} = \vec{\nabla} \times \vec{A}, \qquad \vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}. \tag{1.3}$$

Replacing, the Maxwell equations reduce to four second order equations for the four fields Φ and $\vec{A}$:

$$\Delta\Phi + \frac{\partial}{\partial t}\vec{\nabla} \cdot \vec{A} = -\frac{1}{\epsilon_0}\,\rho, \qquad \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \Delta\vec{A} + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) = \frac{1}{\epsilon_0 c^2}\,\vec{j}. \tag{1.4}$$

Gauge invariance of Maxwell theory is the observation that the electromagnetic fields $\vec{B}$ and $\vec{E}$ stay unchanged if Φ and $\vec{A}$ undergo the transformation

$$\vec{A} \quad\longrightarrow\quad \vec{A}' = \vec{A} - \vec{\nabla}\lambda\,, \qquad\qquad \Phi \quad\longrightarrow\quad \Phi' = \Phi + \frac{\partial\lambda}{\partial t} \tag{1.5}$$

for an arbitrary function of the space-time point $\lambda(t,\vec{x})$. Then, gauge invariance indicates that three functions are sufficient to determine the electromagnetic fields $\vec{E}$ and $\vec{B}$.

For instance, we may rewrite eqs. (1.4)

$$\Box\Phi = \frac{1}{\epsilon_0}\,\rho + \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \vec{\nabla}\cdot\vec{A}\right), \qquad \Box\vec{A} = \frac{1}{\epsilon_0 c^2}\,\vec{j} - \vec{\nabla}\left(\frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \vec{\nabla}\cdot\vec{A}\right), \tag{1.6}$$

where the d'Alembertian operator is

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta, \qquad\qquad \Delta = \vec{\nabla}\cdot\vec{\nabla}. \tag{1.7}$$

Since, under gauge transformation (1.5),

$$\frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \vec{\nabla}\cdot\vec{A} \quad\longrightarrow\quad \frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \vec{\nabla}\cdot\vec{A} + \Box\lambda,$$

we may choose the Lorentz gauge

$$\frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0 \tag{1.8}$$

in which Maxwell equations simplify to

$$\Box\Phi = \frac{1}{\epsilon_0}\,\rho, \qquad\qquad \Box\vec{A} = \frac{1}{\epsilon_0 c^2}\,\vec{j}. \tag{1.9}$$

Maxwell theory of electrodynamics is a relativistic theory: Maxwell equations are covariant under Lorentz transformations with an invariant speed of light c (c is a Lorentz scalar). To write the Maxwell equations in Lorentz-covariant form, we define the following four-vectors:[1]

$$A^\mu(x) = \left(\frac{1}{c}\Phi, \vec{A}\right), \qquad\qquad J^\mu(x) = \frac{1}{\epsilon_0 c^2}\left(c\rho, \vec{j}\right). \tag{1.10}$$

The Lorentz gauge condition is $\partial^\mu A_\mu = 0$ and gauge transformation (1.5) rewrites

$$A^\mu \qquad\longrightarrow\qquad A^\mu + \partial^\mu\lambda. \tag{1.11}$$

It clearly leaves the quantity

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{1.12}$$

[1] Different conventions exist. They coincide in units $c = 1$.

invariant. Since $\Box = \partial^\mu \partial_\mu$, Maxwell equations (1.6) become

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) = \partial_\nu F^{\nu\mu} = J^\mu. \tag{1.13}$$

The identification with the source equations in the second line of equalities (1.1) corresponds finally to

$$E^i = cF^{i0}, \qquad B^i = -\frac{1}{2}\epsilon^{ijk}F^{jk}, \qquad (i,j,k = 1,2,3) \tag{1.14}$$

(E^i and B^i are the three components of $\vec{E}$ and $\vec{B}$) which defines the gauge-invariant electric and magnetic fields. Notice again that Maxwell equations only make sense for a conserved current J^μ:

$$0 = \partial_\mu J^\mu = \frac{1}{\epsilon_0 c^2}\left(\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j}\right). \tag{1.15}$$

In the absence of sources, $J^\mu = 0$, Maxwell equations (1.13) describe electromagnetic waves freely propagating at the speed of light, with only two polarisation states. To see this, consider a plane wave with 4-momentum k^μ and polarisation vector $\epsilon^\mu(k)$:[2]

$$A^\mu(k) = \epsilon^\mu(k)\, e^{ikx}.$$

Maxwell equations impose

$$k^2\, \epsilon^\mu(k) = [k\epsilon(k)]\, k^\mu. \tag{1.16}$$

There are three independent solutions. The first solution has a polarisation vector proportional to k. We may then write

$$A^\mu(k) = \partial^\mu[-if(k)\, e^{ikx}], \qquad \epsilon^\mu(k) = f(k)\, k^\mu.$$

Since A^μ is the derivative ∂^μ of a scalar function, this solution corresponds to vanishing electromagnetic fields, $F^{\mu\nu} = 0$, it can be eliminated by a gauge transformation (1.11), it is unphysical: a potential corresponding to identically zero electromagnetic fields. The other two independent solutions have $k^2 = 0$ (massless waves propagating at speed c) and $k\epsilon(k) = 0$, with $\epsilon(k)$ not proportional to k. With $k^2 = 0$, there is a light-cone frame in which $k = (E, E, 0, 0)$ and then $\epsilon(k) = (0, 0, \epsilon_1, \epsilon_2)$. Hence, the (massless) Maxwell field has only two transverse (to the momentum $\vec{k}$) polarisation states, with helicities ± 1. The third potential solution has been removed by gauge invariance.

It is easy to verify that Maxwell equations $\partial_\nu F^{\nu\mu} = J^\mu$ are the Euler-Lagrange equations of the Maxwell Lagrangian

$$\mathcal{L}_{Max.} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu. \tag{1.17}$$

[2] Since Maxwell equations are linear, we may use a complex plane-wave: real fields are linear combinations of complex fields.

The current vector J^μ is an "external" field verifying $\partial_\mu J^\mu = 0$: $\mathcal{L}_{Max.}$ does not describe any source dynamics. Since $F_{\mu\nu}$ is gauge invariant, the gauge variation of the Lagrangian is

$$\delta_{gauge}\mathcal{L}_{Max.} = -J^\mu \partial_\mu \lambda = -\partial_\mu(\lambda J^\mu)$$

since $\partial_\mu J^\mu = 0$. The action $S = \int d^4x\, \mathcal{L}_{Max}$ is invariant and its Euler-Lagrange equations are then gauge invariant.

1.2 Spinor fields, Dirac Lagrangian

1.2.1 Spinors

Elementary charged particles are mostly spin 1/2 fermions (charged leptons e, μ, τ, quarks). We then need a field and a Lorentz-invariant Lagrangian to describe spin 1/2 particles. In non-relativistic quantum mechanics, the description of spin 1/2 particles uses a doublet of complex wave functions, on which act spin operators $\vec{S}$. These operators have commutation relations

$$[S^i, S^j] = i\,\epsilon^{ijk}\,S^k \tag{1.18}$$

which correspond to the Lie algebra of $SU(2)$ or $SO(3)$ (see chapter 2). In other words, the quantum mechanical description of spin 1/2 particles uses the two-dimensional representation of $SU(2)$.

In a relativistic theory, we need a representation of the Lorentz algebra acting on fields $\psi_a(x)$ with spin 1/2. Infinitesimal Lorentz transformations read

$$\delta\,\psi_a(x) = -\frac{i}{2}\epsilon_{\mu\nu}(M^{\mu\nu})_a{}^b\,\psi_b(x), \tag{1.19}$$

where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ are the six parameters of Lorentz transformations. The generators of the representation $M^{\mu\nu}$ are matrices verifying the Lie algebra relations

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\left(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma}\right). \tag{1.20}$$

One easily verifies that

$$S^1 = M^{23}, \qquad S^2 = M^{31}, \qquad S^3 = M^{12} \tag{1.21}$$

verify relations (1.18) and the Lorentz algebra includes then the spin $SU(2)$ algebra as a subalgebra.

To construct the appropriate Lorentz generators $M^{\mu\nu}$, the first step is to consider the *Clifford algebra*[3]

$$\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}\,\mathbb{I}\,, \qquad \{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu, \tag{1.22}$$

in terms of *Dirac* (or Clifford) *matrices* γ^μ. Their dimension is found by solving relations (1.22): they can be realized in terms of 4×4 matrices and $\mathbb{I}$ is then the identity matrix $\mathbb{I}_4$ in four dimensions. An exemple of solution is

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad i = 1, 2, 3, \tag{1.23}$$

where σ_i are Pauli matrices and I_2 is the two-dimensional identity matrix. This solution is the *Weyl* or *chiral representation.* A representation in terms of 4×4 imaginary matrices is

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \ \gamma^1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}, \ \gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \ \gamma^3 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}. \tag{1.24}$$

Imaginary representations are often called *Majorana representations.* All representations of the Clifford algebra in terms of 4×4 matrices are actually equivalent.[4] Notice that

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0, \qquad \mathrm{Tr}\,\gamma^\mu = 0. \tag{1.25}$$

The first important fact is that, using Clifford algebra (1.22), generators

$$M^{\mu\nu} = \sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{1.26}$$

verify the commutation relations (1.20) of Lorentz algebra. Hence there exists a representation of Lorentz algebra using four-component complex spinors[5]

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}. \tag{1.27}$$

The four-component field $\psi(x)$ is a *Dirac spinor*, with variation

$$\begin{array}{cccl} \psi(x) & \longrightarrow & \psi'(x') & = \psi(x) + \delta\psi(x), \\ & & \delta\psi(x) & = -\frac{i}{2}\,\epsilon_{\mu\nu}\sigma^{\mu\nu}\psi(x) \end{array} \tag{1.28}$$

[3] Also named Dirac algebra. The Clifford algebra has $\delta^{\mu\nu}$ instead of $\eta^{\mu\nu}$, and signs are changed by simply multiplying the corresponding γ^μ by $\pm i$.

[4] Two choices γ^μ and $\tilde{\gamma}^\mu$ are always related by $\tilde{\gamma}^\mu = M^{-1}\gamma^\mu M$, with a single matrix M.

[5] In formula (1.19), a and b take then four values.

when the Lorentz transformation of space-time coordinates $\delta x^\mu = x'^\mu - x^\mu = \eta^{\mu\nu}\epsilon_{\nu\rho}x^\rho$ is applied.

In the Majorana representation (1.24) where γ^μ matrices are imaginary, Lorentz generators are also imaginary. It is then possible to impose a Lorentz-invariant reality condition on the spinor ψ, reducing the number of fields by a factor 1/2. In this way, one defines a *Majorana spinor*. Of course, a similar, but more complicated, constraint exists in all realizations of the γ^μ matrices. Majorana spinors, however, cannot have an electric charge, or any other type of charge.

In quantum field theory, a Dirac spinor describes four particle states: a spin 1/2 particle with mass m (two components) and its antiparticle, with same mass m (two components).[6] The Majorana spinor describes a spin 1/2 particle identical to its antiparticle. It then cannot have any charge, which would necessarily have opposite signs for the particle and the antiparticle.

The second important fact is that there exists a fifth matrix

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{1.29}$$

which verifies

$$\{\gamma_5, \gamma^\mu\} = 0 \qquad (\mu = 0,1,2,3), \qquad \gamma_5{}^\dagger = \gamma_5, \qquad \gamma_5{}^2 = \mathbb{I}_4, \qquad \mathrm{Tr}\,\gamma_5 = 0.$$

In the chiral representation (1.23), γ_5 is diagonal:

$$\gamma_5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}. \tag{1.30}$$

In the Majorana representation (1.24), it is

$$\gamma_5 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}. \tag{1.31}$$

It follows that γ_5 commutes with Lorentz generators,

$$[\gamma_5, \sigma^{\mu\nu}] = 0, \tag{1.32}$$

and it can then used to impose constraints on the spinor field. Concretely, one defines two *chirality projectors*

$$P_L = \frac{1}{2}(\mathbb{I}_4 + \gamma_5), \qquad P_R = \frac{1}{2}(\mathbb{I}_4 - \gamma_5), \tag{1.33}$$

verifying

$$P_L{}^2 = P_L, \qquad P_R{}^2 = P_R, \qquad P_L P_R = P_R P_L = 0, \qquad P_L + P_R = \mathbb{I}_4 \tag{1.34}$$

[6] The field operator $\psi(x)$ destroys a particle or creates an antiparticle.

(a complete set of orthogonal projectors). They commute with Lorentz generators

$$[P_L, \sigma^{\mu\nu}] = [P_R, \sigma^{\mu\nu}] = 0, \tag{1.35}$$

and their eigenvalues are $1, 1, 0, 0$. The *Weyl spinors*

$$\psi_L = P_L\psi = P_L\psi_L, \qquad \psi_R = P_R\psi = P_R\psi_R, \tag{1.36}$$

contain then two complex fields only and transform under Lorentz variations according to

$$\begin{array}{rclclcl} \delta\psi_L &=& -\frac{i}{2}\epsilon_{\mu\nu}\sigma^{\mu\nu}P_L\psi &=& -\frac{i}{2}\epsilon_{\mu\nu}P_L\sigma^{\mu\nu}\psi &=& P_L\delta\psi, \\ \delta\psi_R &=& -\frac{i}{2}\epsilon_{\mu\nu}\sigma^{\mu\nu}P_R\psi &=& -\frac{i}{2}\epsilon_{\mu\nu}P_R\sigma^{\mu\nu}\psi &=& P_R\delta\psi. \end{array} \tag{1.37}$$

The indices L and R stand for *left-handed* and *right-handed chirality*. Each two-component Weyl spinor carries a representation of the Lorentz algebra, it is the basic field to describe in a relativistic field theory spin 1/2 particles. Since the four-component Dirac spinor is the sum $\psi_L + \psi_R$, is carries a reducible representation of Lorentz symmetry. In the chiral representation (1.23) of γ^μ matrices,

$$\psi_L = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix}, \qquad \psi_R = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix}, \tag{1.38}$$

in terms of two-component Weyl spinors χ_L and χ_R.

In quantum field theory, the field operator $\chi_L(x)$ destroys a left-handed particle or creates a right-handed antiparticle. The field $\chi_R(x)$ destroys a right-handed particle or creates a left-handed antiparticle. It is then clear that one cannot impose simultaneously Weyl and Majorana conditions to a Dirac spinor. Firstly, this would reduce the number of particle states to one, which is impossible for a spin 1/2 object. Secondly, Weyl spinors are naturally charged and distinguish then particles and antiparticles.

1.2.2 Mass and spin

The next task is to prove that spinors have spin 1/2. The complete symmetry of special relativity is the Poincaré group or algebra, which includes Lorentz transformations and space-time translations. On a generic field $\phi(x)$, translations act by simply shifting coordinates x^μ by a constant, $\delta x^\mu = a^\mu$:

$$\phi(x) \qquad \longrightarrow \qquad \phi(x+a) = \exp\left(a^\mu \frac{\partial}{\partial x^\mu}\right)\phi(x) = \exp\Big(i\, a^\mu\, P_\mu\Big)\phi(x).$$

The momentum operator, which generates translations, is then[7]

$$P_\mu = -i\frac{\partial}{\partial x^\mu} = -i\partial_\mu\,. \tag{1.39}$$

[7]In non-relativistic quantum mechanics, $\vec{P} = -i\hbar\vec{\nabla}$.

The Poincaré algebra is generated by translations P_μ and Lorentz transformations $M^{\mu\nu}$. Lorentz commutation relations (1.20) are completed by

$$[M^{\mu\nu}, P^\rho] = -i\,\eta^{\mu\rho}P^\nu + i\,\eta^{\nu\rho}P^\mu, \qquad\qquad [P^\mu, P^\nu] = 0. \tag{1.40}$$

This algebra has two Casimir operators I_1 and I_2 which, since (by definition)

$$[I_{1,2}\,, P_\mu] = [I_{1,2}\,, M^{\mu\nu}] = 0,$$

are proportional to the identity on any irreducible representation (Schur's lemma). The first Casimir operator is $I_1 = \eta^{\mu\nu}P_\mu P_\nu = P^2$ and its eigenvalue is m^2, the square of the mass m of the field.[8] Then, for any relativistic field $\phi(x)$,

$$-(P^2 - m^2)\,\phi(x) = (\Box + m^2)\,\phi(x) = 0 \qquad\qquad \text{(Klein-Gordon equation).} \tag{1.41}$$

This is in particular true for the spinor field $\psi(x)$.

The second Casimir operator is more subtle. One introduces the *Pauli-Lubanski vector*

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}, \tag{1.42}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is completely antisymmetric with $\epsilon_{0123} = 1$. Using relations (1.20) and (1.40), one obtains

$$\begin{aligned}
[W_\mu, P_\nu] &= 0, \\
[W_\mu, M_{\nu\rho}] &= -i\left(\eta_{\mu\nu}W_\rho - \eta_{\mu\rho}W_\nu\right), \\
[W_\mu, W_\nu] &= -i\epsilon_{\mu\nu\rho\sigma}P^\rho W^\sigma.
\end{aligned} \tag{1.43}$$

Then, the second Casimir operator is

$$I_2 = W^2 = W^\mu W_\mu, \qquad\qquad [W^2, P^\mu] = [W^2, M^{\mu\nu}] = 0. \tag{1.44}$$

Using the explicit values of $\epsilon_{\mu\nu\rho\sigma}$, another useful form of W^2 can be derived:[9]

$$W^2 = -\frac{1}{2}\left(P^2 M^{\mu\nu}M_{\mu\nu} + 2P^\nu P_\rho M^{\rho\sigma}M_{\sigma\nu}\right). \tag{1.45}$$

Again, each (irreducible) field has a specific eigenvalue under W^2. Altogether, the Poincaré algebra has six mutually commuting operators: P^μ, P^2 and W^2. They can be simultaneously diagonalized, with eigenvalues p^μ, $m^2 = p^2$ and λ_{W^2}.

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[8]Notice that the sign of m^2 could in principle be negative. The field would then describe a tachyon. Such a field is allowed by special relativity but a particle with negative m^2 has never been seen in Nature.

[9]Recall that operators do not in general commute.

Suppose that the field is massive: $p^2 = m^2 > 0$. By a Lorentz transformation, we can choose a rest frame in which $p^\mu = (m, 0, 0, 0)$. This frame is invariant under space rotations generated by M^{12}, M^{23} and M^{31} and with this choice,

$$W^0 = 0, \qquad W^1 = mM^{23} = mS^1, \qquad W^2 = mM^{31} = mS^2, \qquad W^3 = mM^{12} = mS^3$$

[see eqs. (1.21)]. In addition, since

$$[W^i, W^j] = i\epsilon^{ijk} m W^k, \qquad [W^i, W^0] = 0, \qquad (i, j, k = 1, 2, 3),$$

we also recover the spin algebra (1.18). Hence

$$I_2 = W^2 = W^0W^0 - \vec{W}\cdot\vec{W} = -m^2\vec{S}^2 \tag{1.46}$$

with eigenvalue

$$\lambda_{W^2} = -m^2 s(s+1) \qquad (m^2 > 0) \qquad s = 0, 1/2, 1, 3/2, \ldots \tag{1.47}$$

The conclusion is that any relativistic field (*i.e.* any field carrying an irreducible representation of Poincaré algebra) has a mass m and, if $m^2 > 0$, a spin s. This result is true in all frames since m^2 and λ_{W^2} are eigenvalues of Lorentz-invariant operators.

The case of a massless ($m^2 = 0$) field is different since a rest frame does not exist. Instead, we may choose the light-cone frame $p^\mu = (E, E, 0, 0)$, which is invariant under M^{23} only. Since by definition $W^\mu P_\mu = 0$, we can write

$$W_\mu = \lambda p_\mu + (0, 0, W_2, W_3), \qquad W^\mu W_\mu = -(W_2)^2 - (W_3)^2.$$

with $[W_2, W_3] = 0$. Massless[10] particles observed in Nature have $W_2 = W_3 = 0$ and then

$$W_\mu = \lambda p_\mu, \qquad W^2 = 0 \qquad (m = 0). \tag{1.48}$$

The proportionality constant is the *helicity* of the field. In our light-cone frame,

$$W^0 = W^1 = EM^{23} = ES^1 = \vec{p}\cdot\vec{S}, \qquad \lambda = \frac{1}{|\vec{p}|}\vec{p}\cdot\vec{S}. \tag{1.49}$$

Helicity is then the projection of spin along the (spatial) momentum.

We can now return to the proof that a spinor field has spin 1/2. The Lorentz generators are given by eq. (1.26). For a massive field, inserting these Lorentz generators in eq. (1.45) in the rest frame $p_\mu = (m, 0, 0, 0)$ leads to

$$W^2 = -m^2\, s(s+1)\, \mathbb{I}_4 = -\frac{3}{4}m^2\, \mathbb{I}_4, \tag{1.50}$$

[10] Or nearly massless, within experimental accuracies.

and the spin of the field is 1/2.

Notice finally that in the chiral representation (1.23) of γ^μ matrices where Weyl spinors are of the form (1.38), spin operators in the rest frame (1.21) read

$$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}.$$

Each Weyl spinor ψ_L and ψ_R describes then a spin 1/2, and the two components of each Weyl spinor have helicities $+1/2$ and $-1/2$.

1.2.3 Dirac equation and Lagrangian

We need a wave equation for the Dirac spinor $\psi(x)$. Special relativity requires covariance under Lorentz transformations of the field equation and also that solutions of the field equation verify Klein-Gordon equation (1.41). Dirac equation is

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0. \tag{1.51}$$

Solutions also verify

$$\begin{aligned} 0 &= (i\gamma^\mu\partial_\mu + m\,\mathbb{I}_4)(i\gamma^\nu\partial_\nu - m\,\mathbb{I}_4)\,\psi \\ &= \left(-\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu - m^2\,\mathbb{I}_4\right)\,\psi = -(\Box + m^2)\,\psi \end{aligned}$$

which is Klein-Gordon equation. To rewrite Dirac equation in terms of Weyl spinors, use chirality projectors (1.33). Since

$$P_L\gamma^\mu = \gamma^\mu P_R, \qquad\qquad P_R\gamma^\mu = \gamma^\mu P_L,$$

we obtain

$$\begin{aligned} P_L\,(i\gamma^\mu\partial_\mu - m)\,\psi &= i\gamma^\mu\partial_\mu\psi_R - m\psi_L = 0, \\ P_R\,(i\gamma^\mu\partial_\mu - m)\,\psi &= i\gamma^\mu\partial_\mu\psi_L - m\psi_R = 0. \end{aligned} \tag{1.52}$$

Then, to describe a massive field, a four-component Dirac spinor $\psi = \psi_L + \psi_R$ is necessary.[11] In contrast, the description of a massless field is consistent with a single Weyl spinor, with wave equation

$$i\gamma^\mu\partial_\mu\psi_{L,R} = 0. \tag{1.53}$$

To verify the relativistic covariance of Dirac equation, perform a Lorentz transformation of the coordinates,

$$x'^{\,\mu} = \Lambda^\mu{}_\nu x^\nu, \qquad \partial_\mu{}' = \Lambda_\mu{}^\nu\partial_\nu, \qquad \eta_{\rho\sigma} = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\eta_{\mu\nu}, \qquad \Lambda_\mu{}^\nu = \eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^\rho{}_\sigma,$$

[11] There is another possibility: impose a reality condition on a Dirac spinor (the Dirac operator $i\gamma^\mu\partial_\mu - m\mathbb{I}_4$ is real in the Majorana representation). The result is a *Majorana spinor* which however cannot have an electric charge. This possibility will not be further discussed at this stage.

and assume that if $\psi(x)$ is a solution in coordinates x^μ, there exists a linear transformation

$$\psi'(x') = S(\Lambda)\psi(x)$$

such that $\psi'(x')$ is a solution in coordinates x'^μ. We have

$$\begin{aligned}(i\gamma^\mu \partial_\mu{}' - m)\psi'(x') &= (i\gamma^\mu \Lambda_\mu{}^\nu \partial_\nu - m) S(\Lambda)\psi(x) \\ &= S(\Lambda)\Big[iS(\Lambda)^{-1}\gamma^\mu S(\Lambda)\,\Lambda_\mu{}^\nu \partial_\nu - m\Big]\psi(x).\end{aligned}$$

Relativistic covariance is then obtained if

$$S(\Lambda)^{-1}\gamma^\mu S(\Lambda)\,\Lambda_\mu{}^\nu = \gamma^\nu.$$

We now consider an infinitesimal Lorentz transformation of coordinates and of the spinor field:

$$\Lambda_\mu{}^\nu = \delta_\mu^\nu + \epsilon_{\mu\rho}\eta^{\rho\nu}, \qquad S(\Lambda) = \mathbb{I}_4 - \frac{i}{2}\epsilon_{\rho\sigma}M^{\rho\sigma} = \mathbb{I}_4 + \frac{1}{8}\epsilon_{\rho\sigma}[\gamma^\rho, \gamma^\sigma]$$

and $S(\Lambda)^{-1} = \mathbb{I}_4 - \frac{1}{8}\epsilon_{\rho\sigma}[\gamma^\rho, \gamma^\sigma]$. We then need

$$[[\gamma^\rho, \gamma^\sigma], \gamma^\nu] = 4(\gamma^\rho\,\eta^{\nu\sigma} - \gamma^\sigma\,\eta^{\nu\rho}).$$

Since this equation is a consequence of Clifford algebra (1.22), relativistic covariance of Dirac equation is verified.

The Dirac equation is the Euler-Lagrange equation of the Dirac Lagrangian, which depends on ψ and on the conjugate spinor $\overline{\psi}$. This conjugate spinor is such that the quantity $\overline{\psi}\psi$ is Lorentz invariant. Since $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$, we also have $\sigma^{\mu\nu\dagger} = \gamma^0\sigma^{\mu\nu}\gamma^0$ and then, according to eq. (1.19),

$$\delta\psi^\dagger = \frac{i}{2}\epsilon_{\mu\nu}\psi^\dagger\sigma^{\mu\nu\dagger} = \frac{i}{2}\epsilon_{\mu\nu}\psi^\dagger\gamma^0\sigma^{\mu\nu}\gamma^0, \qquad \delta\psi^\dagger\gamma^0 = \frac{i}{2}\epsilon_{\mu\nu}\psi^\dagger\gamma^0\sigma^{\mu\nu}.$$

The *Dirac conjugate* of spinor ψ is then defined as

$$\overline{\psi} = \psi^\dagger\gamma^0. \tag{1.54}$$

One easily verifies that Dirac equation follows from the action principle applied to Lagrangian

$$\mathcal{L}_D(\psi, \overline{\psi}) = \frac{i}{2}\overline{\psi}\gamma^\mu(\partial_\mu\psi) - \frac{i}{2}(\partial_\mu\overline{\psi})\gamma^\mu\psi - m\overline{\psi}\psi. \tag{1.55}$$

Variation with respect to $\overline{\psi}$ gives eq. (1.51), variation wih respect to ψ gives the conjugate equation

$$i\partial_\mu\overline{\psi}\gamma^\mu + m\overline{\psi} = 0. \tag{1.56}$$

A simpler form of the Dirac Lagrangian is

$$\mathcal{L} = i\,\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi. \tag{1.57}$$

Since $\mathcal{L} - \mathcal{L}_D = \frac{i}{2}\partial_\mu(\overline{\psi}\gamma^\mu\psi)$, both Lagrangians lead to identical field equations. The first form $\mathcal{L}_D$ has however the advantage of being explicitly hermitian.

In terms of Weyl spinors,

$$\mathcal{L} = i\overline{\psi_L}\gamma^\mu\partial_\mu\psi_L + i\overline{\psi_R}\gamma^\mu\partial_\mu\psi_R - m\overline{\psi_L}\psi_R - m\overline{\psi_R}\psi_L, \tag{1.58}$$

where

$$\overline{\psi_L} = \psi_L^\dagger\gamma^0 = \overline{\psi}P_R, \qquad\qquad \overline{\psi_R} = \overline{\psi}P_L. \tag{1.59}$$

As already mentioned, both Weyl spinors are coupled by the mass term only.

The Dirac Lagrangian is clearly invariant under phase rotations of the spinor field:

$$\psi \longrightarrow e^{i\alpha}\,\psi, \qquad\qquad \overline{\psi} \longrightarrow e^{-i\alpha}\overline{\psi}. \tag{1.60}$$

The real number α is an arbitrary parameter and the theory has then $U(1)$ symmetry.

The massless theory has a larger symmetry. In particular, phase rotations of ψ_L and Ψ_R are now independent symmetries, and each symmetry has its own conserved Noether current. Consider transformations

$$\begin{array}{lcllcl} \psi_L & \longrightarrow & e^{i\alpha}\,\psi_L, & \qquad \overline{\psi_L} & \longrightarrow & e^{-i\alpha}\,\overline{\psi_L}, \\ \psi_R & \longrightarrow & e^{i\beta}\,\psi_R, & \qquad \overline{\psi_R} & \longrightarrow & e^{-i\beta}\,\overline{\psi_R}, \end{array} \tag{1.61}$$

which are called *chiral transformations*. The conserved chiral currents are then

$$\begin{aligned} J_L^\mu &= \overline{\psi_L}\gamma^\mu\psi_L = \tfrac{1}{2}\,\overline{\psi}\gamma^\mu(1+\gamma_5)\psi \propto \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi_L}\delta\psi_L, \\ J_R^\mu &= \overline{\psi_R}\gamma^\mu\psi_R = \tfrac{1}{2}\,\overline{\psi}\gamma^\mu(1-\gamma_5)\psi \propto \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi_R}\delta\psi_R. \end{aligned} \tag{1.62}$$

The *axial current*

$$J_A^\mu = J_L^\mu - J_R^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi \tag{1.63}$$

verifies, for solutions of Dirac equation,

$$\partial_\mu J_A^\mu = 2im\,\overline{\psi}\gamma_5\psi.$$

It is conserved if the spinor field is massless. For nonzero mass m, only the *vector current*

$$J_V^\mu = J_L^\mu + J_R^\mu = \overline{\psi}\gamma^\mu\psi \tag{1.64}$$

is conserved.

1.3 Electromagnetism of a charged fermion fields

Suppose that the source of the electromagnetic field is a Dirac spinor field $\psi(x)$ with electric charge[12] $-Qe$ ($e = 1.602 \times 10^{-19}$ C is the proton or positron charge, Q is a number). Its free propagation is described by Dirac Lagrangian (1.57)[13]

$$\mathcal{L}_\psi = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \quad \Longrightarrow \quad i\gamma^\mu \partial_\mu \psi = m\psi, \qquad i\partial^\mu \overline{\psi}\gamma_\mu = -m\overline{\psi}, \tag{1.65}$$

with $\overline{\psi} = \psi^\dagger \gamma^0$. The massive Dirac Lagrangian has a conserved Noether current $\sim J^\mu_V$ [eq. (1.64)], related to the global $U(1)$ invariance of $\mathcal{L}_\psi$. We will write this current

$$J^\mu_\psi = eQ\,\overline{\psi}\gamma^\mu\psi \tag{1.66}$$

with Noether charge

$$Q_\psi = \int d^3x\, J^0 = eQ \int d^3x\, \psi^\dagger\psi. \tag{1.67}$$

The Noether current is conserved, $\partial_\mu J^\mu_\psi = 0$, for spinor fields verifying Dirac equation.

Consider now the Lagrangian

$$\begin{aligned} \mathcal{L} &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu_\psi + \mathcal{L}_\psi \\ &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma^\mu\partial_\mu - eQA_\mu\gamma^\mu - m)\psi \end{aligned} \tag{1.68}$$

The field equation of the spinor field is now

$$i\gamma^\mu\partial_\mu\psi = m\psi + eQA_\mu\gamma^\mu\psi, \qquad -i\partial_\mu\overline{\psi}\gamma^\mu = m\overline{\psi} + eQA_\mu\overline{\psi}\gamma^\mu. \tag{1.69}$$

There is an interaction with the electromagnetic field but the current J^μ_ψ stays nevertheless conserved: the new Lagrangian is also invariant under the global symmetry (1.60). The strength of the interaction is given by the constant eQ: it is the electric charge of the fermion described by ψ.

Under gauge transformations however,

$$\delta_{gauge}\mathcal{L} = -eQ\,(\overline{\psi}\gamma^\mu\psi)\,\partial_\mu\lambda = -\partial_\mu(\lambda J^\mu_\psi) + \lambda\,\partial_\mu J^\mu_\psi \tag{1.70}$$

The action is only invariant for spinor fields verifying their field equation, it is not a symmetry of the action functional of arbitrary fields A_μ and ψ.

It however turns out that theory (1.68) has a different symmetry. Promote global symmetry (1.60) with constant parameter α to a local symmetry in which the parameter is a function $\alpha(x)$ of the space-time point. Then,

$$\mathcal{L}_\psi \qquad \longrightarrow \qquad \mathcal{L}_\psi + i\,(\overline{\psi}\gamma^\mu\psi)\,e^{-i\alpha}\partial_\mu e^{i\alpha} = \mathcal{L}_\psi - (\overline{\psi}\gamma^\mu\psi)\,\partial_\mu\alpha.$$

[12] The canonically-quantized spinor field with charge $-Qe$ destroys a particle with charge Qe or creates an antiparticle with charge $-Qe$.

[13] From this point, we use $\hbar = c = 1$.

Comparing with variation (1.70), the choice $\alpha(x) = -eQ\lambda(x)$ leads to an invariant action. As a result, theory (1.68) is invariant under the combined gauge transformation

$$\begin{array}{llllll} A_\mu & \longrightarrow & A_\mu + \partial_\mu \lambda, & & & \\ \psi & \longrightarrow & e^{-ieQ\lambda}\psi, & \overline{\psi} & \longrightarrow & e^{ieQ\lambda}\overline{\psi}. \end{array} \tag{1.71}$$

Observe that

$$D_\mu \psi = (\partial_\mu + ieQA_\mu)\psi \tag{1.72}$$

transforms according to

$$D_\mu \psi \quad \longrightarrow \quad e^{-ieQ\lambda}\, D_\mu \psi. \tag{1.73}$$

Since $D_\mu\psi$ and ψ have the same transformations, $D_\mu\psi$ is called *covariant derivative* of ψ. Similarly,

$$D_\mu \overline{\psi} = (\partial_\mu - ieQA_\mu)\overline{\psi}.$$

With these definitions, theory (1.68) rewrites

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma^\mu D_\mu - m)\psi. \tag{1.74}$$

This is the Lagrangian for quantum electrodynamics (QED) of a fermion with electric charge eQ (the corresponding spinor field ψ carries charge $-eQ$). Its quantization leads to a renormalizable quantum field theory and it is actually the unique gauge-invariant Lagrangian compatible with quantum field theory. Hence imposing the field content (A_μ and ψ), invariance under symmetries (1.71) and compatibility with (perturbative) quantum field theory is sufficient to completely define the Lagrangian and then the dynamics.

1.4 Generalization: QED Lagrangian

The gauge principle works as follows:

1. Postulate the gauge group, the local invariance group of the theory. In the case of quantum electrodynamics, the gauge group is $U(1)$ (local phase rotation).

2. Postulate the transformations of the spin 1/2 and spin 0 fields under the gauge group. In other words, specify in which representation of the gauge group do the spin 1/2 and spin 0 fields transform.

3. Write the most general Lagrangian invariant under the postulated transformations of the fields and compatible with the rules of quantum field theory.

In the case of a single Dirac spinor field with charge $-Qe$, the result is theory (1.74).

Suppose now that we have several Dirac spinor fields ψ_i with charges $-Q_i e$ (think of several quarks with $Q_i = 2/3$ or $Q_i = -1/3$, several charged leptons with charge $Q_i = -1$ and several neutrino fields without electric charge, $Q_i = 0$. Each of these Dirac fields transforms according to

$$\psi_i \qquad \longrightarrow \qquad e^{-ieQ_i\lambda}\,\psi_i \tag{1.75}$$

($\lambda(x)$ is the local parameter of $U(1)$ gauge transformations) under the gauge group of the electromagnetic interaction, which is $U(1)$. The quantum field theory Lagrangian is then

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \textstyle\sum_i \overline{\psi}_i(i\gamma^\mu D_\mu - m_i)\psi_i \\ &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \textstyle\sum_i \overline{\psi}_i(i\gamma^\mu \partial_\mu - m_i)\psi_i \\ &\quad -e\,A_\mu \textstyle\sum_i Q_i\,\overline{\psi}_i\gamma^\mu\psi_i\,.\end{aligned} \tag{1.76}$$

The last line describes the fermion–photon interactions, proportional to the electric charge Q_i (in units of the fundamental proton charge e) of each fermion field. This interaction is of the form

$$-e\,A_\mu\,(\text{gauge potential}) \;\times\; J^\mu\,(\text{conserved current})\,,$$

the conserved electromagnetic current collecting all fermion contributions,

$$J^\mu_{e.m.} = \sum_i Q_i\,\overline{\psi}_i\gamma^\mu\psi_i \qquad\qquad (\partial_\mu J^\mu_{e.m.} = 0). \tag{1.77}$$

The overall strength of the electromagnetic interaction is controlled by the constant e. The quantum field theory perturbative expansion of quantum amplitudes is actually an expansion is powers of the *fine structure constant*

$$\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}. \tag{1.78}$$

In terms of Weyl spinors, theory (1.76) reads

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \textstyle\sum_i[\overline{\psi}_{iL}\,i\gamma^\mu D_\mu\psi_{iL} + \overline{\psi}_{iR}\,i\gamma^\mu D_\mu\psi_{iR}] \\ &\quad - \textstyle\sum_i m_i[\overline{\psi}_{iL}\psi_{iR} + \overline{\psi}_{iR}\psi_{iL}]\end{aligned} \tag{1.79}$$

with

$$D_\mu\psi_{iL} = \partial_\mu\psi_{iL} + ieQ_iA_\mu\,\psi_{iL}\,, \qquad\qquad D_\mu\psi_{iR} = \partial_\mu\psi_{iR} + ieQ_iA_\mu\,\psi_{iR}\,. \tag{1.80}$$

1.4.1 Charge conjugation

Since the spinor field (with charge $-eQ$) describes the particle with charge eQ and the antiparticle with charge $-eQ$, there should exist a transformation $\psi \longrightarrow \psi^C$ such that if the spinor field with charge eQ is a solution of Dirac equation,

$$i\gamma^\mu\partial_\mu\psi - m\psi = eQA_\mu\gamma^\mu\psi,$$

the *charge-conjugate spinor* verifies the Dirac equation with opposite charge:

$$i\gamma^\mu\partial_\mu\psi^C - m\psi^C = -eQA_\mu\psi^C. \tag{1.81}$$

To find the transformation, one observes that for any representation of the γ^μ matrices, there exists a matrix $\mathcal{C}$ such that

$$\mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -\gamma^{\mu\tau} \qquad \text{(and then:} \quad {\gamma_5}^\tau = \mathcal{C}^{-1}\gamma_5\mathcal{C}). \tag{1.82}$$

In the Weyl representation (1.23), one can choose,

$$\mathcal{C} = i\gamma^2\gamma^0, \tag{1.83}$$

with $\mathcal{C} = -\mathcal{C}^{-1} = -\mathcal{C}^\dagger = \mathcal{C}^* = -\mathcal{C}^\tau$. In the Majorana representation (1.24), since $-\gamma^{\mu\tau} = \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$,

$$\mathcal{C} = i\gamma^0. \tag{1.84}$$

Since the spinor ψ verifies $-i\partial_\mu\overline{\psi}\gamma^\mu - m\overline{\psi} = eQA_\mu\overline{\psi}\gamma^\mu$, it also verifies

$$-i\gamma^{\mu\tau}\partial_\mu\overline{\psi}^\tau - m\overline{\psi}^\tau = eQA_\mu\gamma^{\mu\tau}\overline{\psi}^\tau \qquad \Longrightarrow \qquad i\gamma^\mu\mathcal{C}\partial_\mu\overline{\psi}^\tau - m\mathcal{C}\overline{\psi}^\tau = -eQA_\mu\gamma^\mu\mathcal{C}\overline{\psi}^\tau. \tag{1.85}$$

We then define the conjugate spinor as

$$\begin{array}{rclcrcl} \psi^C & = & \mathcal{C}\,\overline{\psi}^\tau \;=\; -\gamma^0\mathcal{C}\psi^*, & \qquad & \overline{\psi}^C & = & \psi^\tau\mathcal{C}, \\ \psi & = & \mathcal{C}\,\overline{\psi}^{C\tau}, & & \overline{\psi} & = & \psi^{C\tau}\,\mathcal{C}. \end{array} \tag{1.86}$$

Since ${P_{L,R}}^* = {P_{L,R}}^\tau = \mathcal{C}^{-1}P_{L,R}\,\mathcal{C}$ and $\gamma^0 = -\mathcal{C}\gamma^{0*}\mathcal{C}^{-1}$,

$$\psi_{L,R} = \mathcal{C}\,{\overline{\psi}^C_{R,L}}^\tau, \qquad\qquad \overline{\psi}_{L,R} = {\psi^C_{R,L}}^\tau\,\mathcal{C}.$$

and, for anticommuting spinor fields,

$$\begin{aligned} i\overline{\psi}_R\gamma^\mu\partial_\mu\psi_R &= -i[\overline{\psi}\gamma^\mu P_R\partial_\mu\psi]^\tau = -i\partial_\mu\psi^\tau {P_R}^\tau\gamma^{\mu\tau}\overline{\psi}^\tau \\ &= i\partial_\mu\psi^\tau\mathcal{C}^{-1}P_R\gamma^\mu\mathcal{C}\overline{\psi}^\tau = -i\psi^\tau\mathcal{C}^{-1}P_R\gamma^\mu\mathcal{C}\partial_\mu\overline{\psi}^\tau + \text{derivative} \\ &= i\overline{\psi}^C_L\gamma^\mu\partial_\mu\psi^C_L + \text{derivative}. \end{aligned} \tag{1.87}$$

We can then always replace the kinetic Lagrangian of the right-handed spinor field by the kinetic term of the charge-conjugate left-handed spinor. We will later on use left-handed Weyl spinors only, with kinetic Lagrangian

$$\mathcal{L} = i\overline{\psi}_L \gamma^\mu \partial_\mu \psi_L + i\overline{\psi}_L^C \gamma^\mu \partial_\mu \psi_L^C - m(\psi_L^{C\tau}\, \mathcal{C}\, \psi_L + \overline{\psi}_L\, \mathcal{C}\, \overline{\psi}_L^{C\tau}) \tag{1.88}$$

for a four-component massive Dirac spinor. Again, the mass term couples ψ_L and ψ_L^C and a single left-handed Weyl spinor only describes a massless spinor, except if ψ is a *Majorana spinor* (without any charge),

$$\psi = \psi^C. \tag{1.89}$$

Chapter 2

On Lie groups and Lie algebras

This chapter gives a very elementary introduction to Lie groups and Lie algebras. It is not meant to replace the substantial literature on this important chapter of mathematics, which includes many texts addressing the needs of physicists.[1] It merely provides minimal tools, introduced with little, if any, mathematical rigour.

2.1 Lie groups

In a field theory, the Noether theorem relates continuous symmetries of the action and conserved quantities. This means that fields of the theory transform in some representation of a Lie group of symmetries, and that these transformations leave invariant the action defining the theory. A Lie group G has in general elements of the form

$$\mathcal{U}\,\cdot\,U(\alpha^A) \tag{2.1}$$

where $\cdot$ is the group product (group internal law), $\mathcal{U}$ belongs to a discrete subset, $\mathcal{U}\in\{\mathbb{I},\mathcal{U}_1,\ldots\}\subset G$ (the discrete set often only includes $\mathbb{I}$, the unit element of the group), and $U(\alpha^A)$, $A=1,\ldots,\dim G$, is an analytic function of a minimal set of independent parameters α^A. We define these parameters such that

$$U(\alpha^A=0)=\mathbb{I}. \tag{2.2}$$

Then, $U(\alpha^A)$ includes all elements of the group connected to the identity.[2] Symbolically, the group law reads

$$U(\alpha^A)\,\cdot\,U(\beta^B)=U(\gamma^C), \tag{2.3}$$

[1]Refs. [1, 2, 4, 3, 5] specifically consider group theory in the context of particle physics theory. More mathematically-oriented texts include refs. [6, 7, 8, 9].

[2]There is a continuous path in G, parameterized by α_A, which links $\mathbb{I}$ to any $U(\alpha_A)$.

it defines the function $\gamma^C(\alpha^A, \beta^B)$, which is differentiable in a Lie group.

In gauge theories and in the Standard Model, the relevant gauge groups are (direct) products of compact[3] simple Lie groups and of $U(1)$ factors. In the simplest case of QED, it is the Lie group $U(1)$. In the Standard Model, it is

$$G_0 = SU(3) \otimes SU(2) \otimes U(1). \tag{2.4}$$

Simple Lie groups are classified. There are four infinite series A_n, B_n, C_n, D_n of classical groups and five exceptional groups E_6, E_7, E_8, F_4 and G_2. The index is the rank of the group: the dimension of the largest abelian subgroup $U(1)^n$. To each entry in the classification corresponds several *real forms*. One of them is compact, the others are non-compact and differ by their maximal compact subgroup. For compact Lie groups, we have:[4]

- A_n $(n \geq 1)$ corresponds to $SU(n+1)$, which can be represented by unitary unimodular $(n+1)$–dimensional matrices: $U^\dagger U = \mathbb{I}$, $\det U = 1$; the number of parameters is $\dim G = (n+1)^2 - 1$.

- B_n $(n \geq 1)$ corresponds to $O(2n+1)$, which can be represented by orthogonal real $(2n+1)$–dimensional matrices: $O^\tau O = \mathbb{I}$; $\dim G = n(2n+1)$. Requiring $\det O = 1$ leads to $SO(2n+1)$.

- C_n $(n \geq 1)$ corresponds to $Sp(2n)$, which can be represented by symplectic real $(2n)$–dimensional matrices:
$$O^\tau \eta_{2n} O = \eta_{2n}, \qquad \eta_{2n} = \begin{pmatrix} 0_n & \mathbb{I}_n \\ -\mathbb{I}_n & 0_n \end{pmatrix};$$
$\dim G = n(2n+1)$.

- D_n $(n \geq 2)$ corresponds to $O(2n)$, which can be represented by orthogonal real $(2n)$–dimensional matrices: $O^\tau O = 1$; $\dim G = n(2n-1)$. Requiring $\det O = 1$ leads to $SO(2n)$. (The Lorentz group $SO(1,3)$ is an example of a non-compact real form of D_2.)

In this list, the realizations of the four classical series correspond to the *fundamental tensor representations*, in terms of linear transformations (matrices) or real or complex vector spaces of minimal dimensions. Each compact simple Lie group has an infinite number of finite-dimensional unitary representations. They can be realized in terms

[3]The parameters α_A take then values in a compact set, like for instance an angle in interval $[0, 2\pi]$.

[4]The terminology used to characterize groups is as follows: U stands for unitary, O for orthogonal, S for special (or unimodular) and Sp for symplectic.

of linear transformations of a complex or real vector space. For instance, as stated earlier, A_n or $SU(n+1)$ has a representation with complex dimension $n+1$ in terms of unitary unimodular transformations of $\mathbb{C}^{n+1}$. Exceptional Lie groups E_6, E_7, E_8, F_4 and G_2 however do not have an intuitive realization.

Unitary groups $U(n)$, which can be represented by $n \times n$ unitary matrices $U^\dagger U = \mathbb{I}_n$ are not simple:

$$U(n) = SU(n) \times U(1).$$

Unitarity implies that the determinant of U is a phase, $\det U = e^{i\Delta}$. We can then write

$$U = e^{i\Delta/n}\mathbb{I}_n \cdot \widetilde{U}, \qquad\qquad \det \widetilde{U} = 1.$$

The first factor is an element of $U(1)$ and $\widetilde{U} \in SU(n)$. For $O(n)$ and $Sp(2n)$ groups, $\det O = \pm 1$ and only elements with $\det O = 1$ are in the part connected to the identity element. Since similar arguments hold for exceptional groups, elements connected to the identity of all simple Lie groups are necessarily unimodular.

The fact that Lie group elements connected to the identity $U(\alpha_A)$ are analytic functions of the parameters allows to write a Taylor expansion, which formally reads

$$\begin{aligned} U(\alpha^A) &= \sum_{n\geq 0} \frac{1}{n!}\,\alpha^{A_1}\ldots\alpha^{A_n}\left[\frac{\partial}{\partial\alpha^{A_1}}\cdots\frac{\partial}{\partial\alpha^{A_n}}U(\alpha^B)\right]_{\alpha^A=0} \\ &\equiv \exp\left[\alpha^A\,\frac{\partial}{\partial\alpha^A}\bigg|_{\alpha^A=0}\right]U(\alpha^B). \end{aligned} \tag{2.5}$$

This expansion is the root of the relation between Lie groups and Lie algebras.

2.2 The Lie algebra of a Lie group

The elements connected to the identity $\mathbb{I}$ of a Lie group have a Taylor expansion (2.5) in powers of the parameters. We may write, for small parameters,

$$U(\alpha^A) = \mathbb{I} + i\,\alpha^A T_A + \mathcal{O}(\alpha^A\alpha^B), \qquad\qquad T_A = -i\,\frac{\partial}{\partial\alpha^A}U(\alpha^B)\bigg|_{\alpha^B=0}. \tag{2.6}$$

Comparing with expansion (2.5), we write the elements connected to the identity as an exponential

$$U(\alpha^B) = e^{\Lambda(\alpha^B)}, \qquad\qquad \Lambda(\alpha^B) = i\alpha^A\,T_A. \tag{2.7}$$

The $\Lambda(\alpha^B)$ belong to a vector space, the *Lie algebra* of the Lie group, and the set $\{T_A, A = 1,\ldots,\dim G\}$ gives a basis of the Lie algebra. The basis elements T_A are the *generators* of the Lie algebra. They are not uniquely defined: any basis of the vector space provides a set of generators. Since, for all simple Lie groups, $U(\alpha^A)$ is unimodular, generators of the corresponding Lie algebras are always traceless,

$$\det U(\alpha^A) = 1 \qquad \forall \alpha^A \qquad \Longrightarrow \qquad \mathrm{Tr}\, T_A = 0 \qquad \forall A. \tag{2.8}$$

Writing $U(\alpha^A) = \exp(i\alpha^A T_A)$, we have used the convention (followed in general by physicists[5]) that Lie algebras of compact groups have finite-dimensional unitary representations with hermitian generators:

$$U(\alpha^A)^{-1} = e^{-i\alpha^A T_A} = U(\alpha^A)^\dagger = e^{-i\alpha^A {T_A}^\dagger} \qquad \Longrightarrow \qquad T_A = {T_A}^\dagger \qquad \forall A. \tag{2.9}$$

Hence, to each Lie group corresponds a single Lie algebra, obtained from the Taylor expansion of the part of the group connected to the identity. The converse is not true: discrete elements of the Lie group [the elements $\mathcal{U}$ in eq. (2.1)] cannot in general be written as the exponential of an element of the Lie algebra. Hence, the Lie algebra does not always suffice to reconstruct, by exponentiation, the whole Lie group and several Lie groups may have the same Lie algebra. However, Lie groups having the same Lie algebra differ by discrete elements. These are not essential in the context of field theories since Noether theorem only refers to continuous symmetries.[6] For instance, orthogonal groups $O(n)$ and their unimodular versions $SO(n)$ have identical Lie algebras. For compact simple Lie groups, we have the following isomorphisms of the corresponding Lie algebras:

$$\begin{array}{ll} A_1 \sim B_1 \sim C_1, & SU(2) \sim SO(3) \sim Sp(2), \\ B_2 \sim C_2, & SO(5) \sim Sp(4), \\ D_2 \sim A_1 \times A_1, & SO(4) \sim SU(2) \times SU(2), \\ D_3 \sim A_3, & SO(6) \sim SU(4). \end{array} \tag{2.10}$$

And of course $O(2)$, $SO(2)$ and $U(1)$ have the same one-dimensional Lie algebra.

The characterization of the generators of Lie algebras of simple compact classical groups, in the fundamental matrix representations, is straightforward:

$$\begin{array}{rrcllcl} A_n: & U^\dagger U & = & \mathbb{I} & \Longrightarrow & T_A & = & {T_A}^\dagger, \\ B_n, D_n: & O^\tau O & = & \mathbb{I} & \Longrightarrow & T_A & = & -{T_A}^\tau, \\ C_n: & O^\tau \eta_{2n} O & = & \eta_{2n} & \Longrightarrow & T_A & = & \eta_{2n} {T_A}^\tau \eta_{2n}. \end{array} \tag{2.11}$$

[5] Mathematicians in general use the opposite convention, with antihermitian generators for compact symmetries, and with $U(\alpha^A) = \exp(\alpha^A T_A)$.

[6] But discrete symmetries may have physical significance. For instance, parity P is not a symmetry of weak interactions. It is an element of $O(3)$ but, since its determinant is -1, it is not an element of $SO(3)$. Both are rotation groups with the same Lie algebra, but $O(3)$ is not a symmetry of particle physics, while $SO(3)$ is, as part of the Lorentz group.

These relations follow from expansion (2.6), for arbitrarily small values of the parameters α^A. They also illustrate how the use of the Lie algebra brings simplification: while conditions defining Lie group elements are quadratic matrix equations, the corresponding Lie algebra equations are *linear*.

2.2.1 Commutation relations, structure constants

The group internal law can be translated into an internal operation on the Lie algebra:

$$U(\alpha^A)\,\cdot\,U(\beta^B) = U(\gamma^C) \qquad \Longrightarrow \qquad e^{i\alpha^A T_A} e^{i\beta^B T_B} = e^{i\gamma^C T_C}. \tag{2.12}$$

The Baker-Campbell-Hausdorff formula indicates that

$$e^a\,e^b = e^c, \qquad\qquad c = a + b + \frac{1}{2}[a,b] + \frac{1}{12}[a,[a,b]] - \frac{1}{12}[b,[a,b]] + \ldots, \tag{2.13}$$

where $[a,b] = ab - ba$ and the dots replace an infinite sum of higher-order commutators. If $[a,b] = 0$, $c = a + b$. The group law imposes then that if $a = i\alpha^A T_A$ and $b = i\beta^B T_B$, then $c = i\gamma^C T_C$ and this is obtained if

$$[T_A, T_B] = i f_{AB}{}^C\, T_C, \qquad\qquad \forall\, A, B. \tag{2.14}$$

The real *structure constants* $f_{AB}{}^C = -f_{BA}{}^C$ completely encode the group law, up to (discrete) elements of the Lie group which are not connected to the identity. Their values depend on the choice of the generators, *i.e.* on the choice of a basis of the Lie algebra. For instance, the group law (2.12) also indicates that

$$U(\alpha^A)^{-1} = U(-\alpha^A).$$

Then, generators $-T_A$ can be used instead of T_A as a basis of the Lie algebra of a given group and changing the sign of generators also changes the sign of the structure constants.

The commutation relations (2.14), which define the internal operation of the Lie algebra, trivially verify Jacobi identity

$$[[T_A, T_B], T_C] + [[T_B, T_C], T_A] + [[T_C, T_A], T_B] = 0. \tag{2.15}$$

The general definition of a Lie algebra is actually a vector space which admits a bilinear internal operation verifying

$$[A, B] = -[B, A] \quad \text{(antisymmetry)},$$
$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0 \quad \text{(Jacobi)}$$

for all its elements.[7] The operation is not associative and in general not abelian. It can be represented as the commutator $[A, B] = AB - BA$ of two linear transformations, but other realizations exist.

The following terminology applies:

- An abelian Lie algebra has vanishing structure constants: $[A, B] = 0$, $\forall A, B$.
- An *invariant subalgebra* $K \subset L$ is preserved by the commutator: $[A, B] \in K$, $\forall A \in K$, $\forall B \in L$.
- For a given Lie algebra, the set of elements which commute with all elements of the Lie algebra is necessarily an abelian subalgebra. It is the *center* of the Lie algebra.
- A *semi-simple* Lie algebra does not have an invariant abelian subalgebra.
- A *simple* Lie algebra does not have an invariant subalgebra.

Semi-simple algebras are direct sums of simple algebras and, as stated earlier, simple Lie algebras follow the same classification as simple Lie groups.

2.2.2 The Cartan-Killing metric

Inserting the commutation relations (2.14) into Jacobi identities (2.15) leads to

$$f_{AB}{}^D f_{CD}{}^E + f_{BC}{}^D f_{AD}{}^E + f_{CA}{}^D f_{BD}{}^E = 0 \tag{2.16}$$

for all values of A, B, C, E.[8] The symmetric *Cartan-Killing metric* is then defined by

$$g_{AB} = -f_{AC}{}^D f_{BD}{}^C = g_{BA}. \tag{2.17}$$

We then also introduce

$$C_{ABC} = f_{AB}{}^D \, g_{DC} \, . \tag{2.18}$$

Using Jacobi identity,

$$\begin{aligned} C_{ABC} + C_{ACB} &= f_{AB}{}^D \, g_{DC} + f_{AC}{}^D \, g_{DB} \;=\; -f_{AB}{}^D \, f_{DE}{}^F \, f_{CF}{}^E - f_{AC}{}^D \, f_{DE}{}^F \, f_{BF}{}^E \\ &= f_{EA}{}^D \, f_{DB}{}^F \, f_{CF}{}^E + f_{EA}{}^D \, f_{DC}{}^F \, f_{BF}{}^E \\ &\quad + f_{BE}{}^D \, f_{DA}{}^F \, f_{CF}{}^E + f_{CE}{}^D \, f_{DA}{}^F \, f_{BF}{}^E \;=\; 0 \end{aligned}$$

[7] A statement equivalent to the antisymmetry condition is $[A, A] = 0, \forall A$ since this implies $[A + B, A + B] = [A, B] + [B, A] = 0$.

[8] This also implies $f_{AB}{}^D f_{DC}{}^C = 0$.

and the constants C_{ABC} are antisymmetric in their three indices. For a general Lie algebra, the Cartan-Killing metric may be degenerate (it obviously vanishes for an abelian algebra). For example,

$$[A, B] = i\,C, \qquad\qquad [A, C] = [B, C] = 0$$

verifies Jacobi identity and defines then a three-dimensional Lie algebra with a single nonzero structure constant ${f_{AB}}^C = 1$ and with identically zero Cartan-Killing metric and f_{ABC}.[9] The metric g_{AB} is non-degenerate ($\det g \neq 0$) if and only if the Lie algebra is semi-simple. It has then an inverse

$$g^{AB} g_{BC} = \delta^A_C. \tag{2.19}$$

2.3 Lie algebra representations

To define a representation R of the Lie algebra with commutation relations (2.14), we need:

- A vector space $\mathcal{V}$ with (complex or real) dimension $\dim R$ and its linear transformations.
- A set $\{T^R_A\}$ of linear transformations of the vector space verifying relations (2.14), $[T^R_A, T^R_B] = i{f_{AB}}^C\, T^R_C$.

The representation R is *reducible* if there is a basis of $\mathcal{V}$ in which all generators T^R_A are block-diagonal. If such a basis does not exist, it is *irreducible*. An irreducible representation (IR) is often simply characterized by its dimension $\dim R$.[10] The generators are then realized as linear operators in $\mathcal{V}$, as matrices with dimension $\dim R$. Then, if ϕ_i, $i = 1, \ldots, \dim R$, is a vector in $\mathcal{V}$, its infinitesimal variation under the Lie algebra is

$$\delta\phi_i = i\alpha^A {(T^R_A)_i}^j\, \phi_j. \tag{2.20}$$

Non-compact simple Lie algebras have a maximal compact subalgebra. In general, we can split the generators according to $\{T^R_A\} = \{T^R_a, \widehat{T}^R_k\}$ with

$$\begin{array}{lllllll} T^R_a &=& {T^R_a}^\dagger, & [T^R_a, T^R_b] &=& i{f_{ab}}^c\, T^R_c, & \text{(compact subalgebra)}, \\ \widehat{T}^R_k &=& -{\widehat{T}^R_k}{}^\dagger, & [\widehat{T}^R_k, \widehat{T}^R_l] &=& i{f_{kl}}^a\, T^R_a, & \text{(non-compact generators)}, \\ & & & [T^R_a, \widehat{T}^R_k] &=& i{f_{ak}}^l\, \widehat{T}^R_l. & \end{array} \tag{2.21}$$

[9] Since C commutes with all elements of the Lie algebra, it is a center element which can be taken proportional to $\mathbb{I}$. Replacing $\{A, B, C\}$ by $\{Q, P, \hbar\mathbb{I}\}$ leads then to $[Q, P] = i\hbar\mathbb{I}$ which is the (one-dimensional) *Heisenberg algebra* of quantum mechanics.

[10] But there are Lie algebras possessing several inequivalent IR with same dimensions.

(Other commutators are zero[11]). The last equation indicates that non-compact generators transform in a representation of the compact subalgebra. Hermitian (anti-hermitian) generators have real (imaginary) eigenvalues. One can then certainly choose

$$\mathrm{Tr}(T_A^R T_B^R) = T(R)\,\eta_{AB}, \qquad \eta = \begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & -\mathbb{I}_q \end{pmatrix}, \qquad p+q = \dim G, \tag{2.22}$$

where p and q are respectively the number of compact and non-compact generators of the Lie algebra. For a compact Lie algebra,

$$\mathrm{Tr}(T_A^R T_B^R) = T(R)\,\delta_{AB}. \tag{2.23}$$

The number $T(R)$ is the *quadratic (Dynkin) index* of the representation.

In matrix notation, the Lie algebra variation (2.20) reads

$$\delta\phi = i\alpha^A T_A^R \phi\,, \qquad \delta\phi^\dagger = -i\alpha^A \phi^\dagger T_A^R\,, \qquad \delta(\phi^\dagger\phi) = 0 \tag{2.24}$$

and also

$$\delta(\phi^\dagger T_A^R \phi) = i\alpha^B\,\phi^\dagger [T_A^R, T_B^R]\phi = i\alpha^B\, i f_{AB}{}^C\,(\phi^\dagger T_C^R \phi). \tag{2.25}$$

The conjugate quantities $\phi^\dagger$ transform in the representation $\overline{R}$, conjugate to R, with generators $-T_A^R$. Hence, $\phi^\dagger\phi$ is an invariant and we will see in the next subsection that the quantities $\phi^\dagger T_R^A \phi$ transform in the adjoint representation of the Lie algebra.

In quantum field theories, the vector space $\mathcal{V}$ carrying the representation is a space of scalar or spinor fields $\phi_i(x)$ with dimension $i = 1, \ldots, \dim R$. The fields transform according to eqs. (2.20) or (2.24) under the Lie algebra of the symmetry group of the theory.

2.3.1 The adjoint representation

Consider linear operators on the real vector space $\mathbb{R}^{\dim G}$ and define the following generators ($\dim G \times \dim G$ matrices):

$$(T_A^{AdjG})_B{}^C = -i f_{AB}{}^C. \tag{2.26}$$

Using Jacobi identities (2.16), the Lie algebra is verified:

$$\begin{aligned}
[T_A^{AdjG}, T_B^{AdjG}]_C{}^D &= (T_A^{AdjG})_C{}^E (T_B^{AdjG})_E{}^D - (T_B^{AdjG})_C{}^E (T_A^{AdjG})_E{}^D \\
&= -f_{AC}{}^E f_{BE}{}^D - f_{CB}{}^E f_{AE}{}^D \\
&= -f_{AB}{}^E f_{CE}{}^D \;=\; i f_{AB}{}^E (T_E^{dimG})_C{}^D.
\end{aligned} \tag{2.27}$$

[11] Notice that this algebra is compatible with real structure constants.

The Lie algebra of a group G always admits then an *adjoint representation* with (real) dimension $\dim_{AdjG} = \dim G$, in terms of linear transformations of $\mathbb{R}^{\dim G}$. The matrix elements of its generators can be naturally defined from the structure constants [eq. (2.26)]. Quantities $x_A \in \mathbb{R}^{\dim G}$ transform in the adjoint representation if their Lie algebra variation is

$$\delta x_A = i\alpha^B (T_B^{AdjG})_A{}^C\, x_C = \alpha^B\, f_{BA}{}^C\, x_C\,, \tag{2.28}$$

as in eq. (2.25). The equation $C_{ABC} + C_{ACB} = 0$ acquires then another significance:

$$C_{ABC} + C_{ACB} = f_{AB}{}^D\, g_{DC} + f_{AC}{}^D\, g_{DB} = i(T_A^{AdjG})_B{}^D\, g_{DC} + i(T_A^{AdjG})_C{}^D\, g_{BD} = 0.$$

Hence, the Lie algebra variation of the symmetric tensor g_{AB} vanishes: the Cartan-Killing metric is an invariant tensor.

For a compact Lie algebra, hermiticity of the generators corresponds to antisymmetry

$$(T_A^{AdjG})_B{}^C = -(T_A^{AdjG})_C{}^B \qquad \Longleftrightarrow \qquad f_{AB}{}^C = -f_{AC}{}^B \tag{2.29}$$

and

$$g_{AB} = -f_{AC}{}^D f_{BD}{}^C = \mathrm{Tr}(T_A^{AdjG} T_B^{AdjG}) = T(AdjG)\,\delta_{AB}. \tag{2.30}$$

2.3.2 Indices, Casimir operators

In this paragraph we consider a generic representation R of a simple Lie algebra, with generators T_A^R normalized with $\mathrm{Tr}(T_A^R T_B^R) = T(R)\,\eta_{AB}$, as in eq. (2.22). If the Lie algebra is compact, $\eta_{AB} = \delta_{AB}$. Then,

$$g_{AB} = \mathrm{Tr}(T_A^{AdjG} T_B^{AdjG}) = T(AdjG)\,\eta_{AB}, \qquad g^{AB} = T(AdjG)^{-1}\eta^{AB}, \qquad \eta^{AB} = \eta_{AB}.$$

Define

$$I_2(R) = \eta^{AB}\, T_A^R\, T_B^R. \tag{2.31}$$

Since η^{AB} is an invariant tensor and since

$$f_{ABC} \equiv f_{AB}{}^D \eta_{DC} = T(AdjG)^{-1}\, C_{ABC} \tag{2.32}$$

is completely antisymmetric, we have:

$$\begin{aligned} [T_A^R, I_2(R)] &= i\eta^{BC}\Big(f_{AB}{}^D T_D^R T_C^R + f_{AC}{}^D T_B^R T_D^R\Big) \\ &= i f_{ABE}\, \eta^{BC}\eta^{ED}\Big(T_D^R T_C^R + T_C^R T_D^R\Big) \;=\; 0. \end{aligned}$$

Since $I_2(R)$ commutes with all elements of the Lie algebra, it is a Casimir operator which can be chosen (Schur's lemma) proportional to the identity for any irreducible representation R:

$$I_2(R) = C(R)\,\mathbb{I}. \tag{2.33}$$

Taking the trace,

$$\mathrm{Tr}\, I_2(R) = \dim R \cdot C(R) = \eta^{AB}\ \mathrm{Tr}(T_A^R T_B^R) = \dim G \cdot T(R)$$

and then

$$C(R) = \frac{\dim G}{\dim R}\, T(R), \qquad\qquad C(G) \equiv C(AdjG) = T(AdjG). \tag{2.34}$$

The number $C(R)$ is the quadratic Casimir number of representation R, $C(G)$ is the quadratic Casimir number of the Lie algebra. The following useful properties hold:

- A representation R is reducible if it is the sum of irreducible representations[12]: $R = R_1 \oplus R_2 \oplus \ldots \oplus R_k$. Then,

$$T(R) = T(R_1) + T(R_2) + \ldots + T(R_k), \tag{2.35}$$

  since the generators can be chosen block-diagonal.

- With two representations R_1 and R_2 (on vector spaces $\mathcal{V}_1$ and $\mathcal{V}_2$), we can always obtain another representation $R = R_1 \otimes R_2$ in terms of linear transformations acting on the tensor product $\mathcal{V}_1 \otimes \mathcal{V}_2$. Then, $\dim R = \dim R_1 \cdot \dim R_2 = \dim \mathcal{V}_1 \cdot \dim \mathcal{V}_2$ and

$$T(R) = \dim R_1 \cdot T(R_2) + \dim R_2 \cdot T(R_1). \tag{2.36}$$

The normalization of all $T(R)$ follows from a single choice, for instance the choice of the structure constants. A more common procedure is, for a given Lie algebra, to use the index of the fundamental tensor representation to define the normalization. For compact simple Lie algebras, we use the following conventions:

Lie algebra	Fundamental IR	Index	Casimir number
$G = SU(N)$	N	$T(N) = 1/2$	$C(SU(N)) = N$
$G = SO(N)$	N	$T(N) = 1$	$C(SO(N)) = N - 2$
$G = Sp(2N)$	$2N$	$T(2N) = 1$	$C(Sp(2N)) = 2N + 2$
$G = E_6$	27	$T(27) = 3$	$C(E_6) = T(78) = 12$
$G = E_7$	56	$T(56) = 6$	$C(E_7) = T(133) = 18$
$G = E_8$	248	$T(248) = 30$	$C(E_8) = T(248) = 30$
$G = F_4$	26	$T(26) = 3$	$C(F_4) = T(52) = 9$
$G = G_2$	7	$T(7) = 1$	$C(G_2) = T(14) = 4$

[12] In the sense of a (direct) sum of vector spaces. A basis of the vector space of R is the union of basis of the vector spaces of R_1, R_2, ..., R_k.

The Dynkin index $T(R)$ is then an integer or half an integer for all IR's.[13]

This construction of the quadratic index and Casimir operator can be extended in principle to higher orders: for the adjoint representation, the number of independent Casimir operators equals the rank of the Lie algebra. Suppose that we have two tensors $U_{A\ldots B}$ and $W_{C\ldots D}$, transforming in the adjoint representation,

$$\delta\, U_{AB\ldots C} = i\alpha^D \Big(f_{DA}{}^E U_{EB\ldots C} + f_{DB}{}^E U_{AE\ldots C} + \ldots + f_{DC}{}^E U_{AB\ldots E} \Big), \tag{2.37}$$

and similarly for $W_{C\ldots D}$. Then, multiplying by the invariant η^{AB} creates invariant "contractions" of indices. Consider for instance

$$U_{C\ldots DE\ldots F} = \eta^{AB}\, U_{AC\ldots D}\, W_{BE\ldots F}.$$

Its Lie algebra variation is

$$\begin{aligned} \delta\, U_{C\ldots DE\ldots F} &= \eta^{AB}\, \delta(U_{AC\ldots D}\, W_{BE\ldots F}) \\ &= i\alpha^G\, \eta^{AB}\, (f_{GA}{}^H\, U_{HC\ldots D}\, W_{BE\ldots F} + f_{GB}{}^H\, U_{AC\ldots D}\, W_{HE\ldots F}) + \ldots \\ &= i\alpha^G\, f_{GAK}\, \eta^{AB}\eta^{HK} (U_{HC\ldots D}\, W_{BE\ldots F} + U_{BC\ldots D}\, W_{HE\ldots F}) + \ldots \end{aligned}$$

The parenthesis is symmetric in HB while f_{GAK} is antisymmetric in AK. Hence, the explicit variation actually cancels and only the dots, which replace the variations of tensor indices $C\ldots DE\ldots F$, survive. Tensors in the adjoint representation can be obtained from symmetric traces of generators[14] in, for instance, the fundamental representation R_f:

$$d^{(k)}_{A_1\ldots A_k} = \mathrm{Tr}\Big(T^{R_f}_{(A_1} T^{R_f}_{A_2} \ldots T^{R_f}_{A_k)} \Big), \qquad d^{(k)\,A_1\ldots A_k} = \eta^{A_1B_1}\ldots\eta^{A_kB_k}\, d^{(k)}_{B_1\ldots B_k}. \tag{2.38}$$

Generic Casimir operators for representation R are then of the form

$$I_k(R) = c\, d^{(k)\,A_1A_2\ldots A_k}\, T^R_{A_1} T^R_{A_2} \ldots T^R_{A_k}\,, \tag{2.39}$$

with a conventional constant c. The quadratic Casimir operator (2.31) uses $d^{(2)\,AB} = \eta^{AB}$.

At third order for instance, the structure constants are obtained from

$$\begin{aligned} &\mathrm{Tr}\Big(T^R_A[T^R_B, T^R_C]\Big) = \mathrm{Tr}(T^R_B[T^R_C, T^R_A]) = \mathrm{Tr}(T^R_C[T^R_A, T^R_B]) \\ &\quad = iT(R) f_{BC}{}^D \eta_{DA} = iT(R) f_{CA}{}^D \eta_{DB} = iT(R) f_{AB}{}^D \eta_{DC} = iT(R)\, f_{ABC}. \end{aligned} \tag{2.40}$$

[13] In the literature, various conventions exist for the Dynkin index, which is then always *proportional to* $T(R)$.

[14] Antisymmetric traces can be reduced to lower orders using the Lie algebra commutation rules.

Similarly, we may consider the symmetric trace:

$$d^R_{ABC} = T(R)^{-1}\ \mathrm{Tr}\Big(T^R_A\{T^R_B, T^R_C\}\Big) = T(R)^{-1}\ \mathrm{Tr}\Big(T^R_A T^R_B T^R_C + T^R_A T^R_C T^R_B\Big). \qquad (2.41)$$

The quantity d^R_{ABC} is the *anomaly coefficient* of irreducible representation R. It is particularly important in gauge field theories with Weyl fermion fields transforming in representation R: a nonzero d^R_{ABC} indicates the existence of a quantum anomaly which destroys the consistency of the theory. Hence, absence of anomaly is required and this amounts to choose only representations R for which all d^R_{ABC} vanish. Since gauge field theories have compact symmetry groups, generators are hermitian and d^R_{ABC} is real. However, if the representation is real, the generators are imaginary and then $d^R_{ABC} = 0$. One can show that only complex representations of $SU(N)$, $N \geq 3$, Lie algebras can have nonzero anomaly coefficients. This can be understood from the following observation: the coefficients d^R_{ABC} are Clebsch-Gordan coefficients for

$$(\ AdjG \quad \otimes \quad AdjG \quad \otimes \quad AdjG\ )_{\text{symmetric}} \qquad \Longrightarrow \qquad 1$$

(1 is the singlet, one-dimensional representation with generators $T^1_A = 0$) or

$$(\ AdjG \quad \otimes \quad AdjG\ )_{\text{symmetric}} \qquad \Longrightarrow \qquad AdjG.$$

All Lie algebras have an adjoint representation in the antisymmetric part of $AdjG \otimes AdjG$ and the structure constants are the Clebsch-Gordan coefficients. But only $SU(N)$ also admits a second adjoint representation, in the symmetric part of the product. For $SU(N)$ one then defines

$$d^R_{ABC} = A(R)\, d_{ABC}, \qquad\qquad d_{ABC} = T(N)^{-1}\ \mathrm{Tr}\Big(T^N_A\{T^N_B, T^N_C\}\Big), \qquad (2.42)$$

where N is the fundamental representation (complex dimension N) and $A(N) = 1$. Formulæ similar to (2.35) and (2.36) apply for the anomaly coefficients $A(R)$, which can also be obtained by calculating $\mathrm{Tr}(T^{R^3}_A)/\,\mathrm{Tr}(T^{N^3}_A)$ for any given generator. Notice that

$$\eta^{AB} d^R_{ABC} = 2\,T(R)^{-1}\ \mathrm{Tr}\Big(T^R_C \eta^{AB} T^R_A T^R_B\Big) = 2\frac{C(R)}{T(R)}\,\mathrm{Tr}(T^R_C) = 0. \qquad (2.43)$$

In general, we may write:

$$\begin{array}{rcl} T^R_A T^R_B &=& \frac{1}{2}\{T^R_A, T^R_B\} + \frac{1}{2}[T^R_A, T^R_B], \qquad\qquad [T^R_A, T^R_B] \;=\; i\, f_{AB}{}^C T^R_C, \\ \{T^R_A, T^R_B\} &=& 2\,\frac{T(R)}{\dim R}\,\eta_{AB}\,\mathbb{I} + M^R_{AB} + A(R) d_{ABD}\eta^{DC} T^R_C, \\ M^R_{AB} &=& M^R_{BA}, \qquad \mathrm{Tr}\, M^R_{AB} \;=\; 0, \qquad \eta^{AB} M^R_{AB} \;=\; 0, \\ \mathrm{Tr}(T^R_A M^R_{BC}) &=& 0. \end{array} \qquad (2.44)$$

Hermiticity of the generators implies $M^R_{AB} = {M^R_{AB}}^\dagger$.

For the fundamental representation N of $SU(N)$, for instance, M^R_{AB} vanishes: it is an hermitian traceless $N \times N$ matrix, and then an element of the Lie algebra of $SU(N)$, which never verifies the last condition. Then,

$$\{T^N_A, T^N_B\} = \frac{1}{N}\,\delta_{AB}\,\mathbb{I} + d_{ABC}T^N_C \tag{2.45}$$

since $T(N) = 1/2$ and $A(N) = 1$. All information on $T^N_A T^N_B$ is in the structure constants and in the tensor d_{ABC}.

For the adjoint representation of $SU(2)$, for which ${(T^{AdjG}_A)_B}^C = -i\epsilon_{ABC}$, the index is $T(AdjG) = 2$, $\dim R = 3$ and $d_{ABC} = 0$,

$$(M_{AB})_{CD} = \frac{2}{3}\delta_{AB}\delta_{CD} - \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}. \tag{2.46}$$

In general, for the adjoint representation, using $A(AdjG) = 0$,

$${\{T^{AdjG}_A, T^{AdjG}_B\}_C}^E = -{f_{AC}}^D{f_{BD}}^E - {f_{BC}}^D{f_{AD}}^E = 2\,\frac{C(G)}{\dim G}\,\eta_{AB}\,\delta^E_C + {(M^{AdjG}_{AB})_C}^E \tag{2.47}$$

with

$${(M^{AdjG}_{AB})_C}^E = -{f_{AC}}^D{f_{BD}}^E - {f_{BC}}^D{f_{AD}}^E - 2\,\frac{C(G)}{\dim G}\,\eta_{AB}\,\delta^E_C. \tag{2.48}$$

Then, firstly,

$$\mathrm{Tr}(M^{AdjG}_{AB}) = {(M^{AdjG}_{AB})_C}^C \;=\; -{f_{AC}}^D{f_{BD}}^C - {f_{BC}}^D{f_{AD}}^C - 2\,C(G)\,\eta_{AB} = 0.$$

Secondly, since $\eta^{AB}{f_{BC}}^D{f_{AD}}^E = -C(G)\delta^E_C$,

$$\eta^{AB}{(M^{AdjG}_{AB})_C}^E = -\eta^{AB}{f_{BC}}^D{f_{AD}}^E - \eta^{AB}{f_{AC}}^D{f_{BD}}^E - 2\,C(G)\,\delta^E_C = 0.$$

Thirdly, with ${f_{AB}}^B = 0$ and $A(AdjG) = 0$,

$$\begin{aligned}
\mathrm{Tr}\Big(T^{AdjG}_F M_{AB}\Big) &= {(T^{AdjG}_F)_E}^C\Big({f_{CB}}^D{f_{AD}}^E + {f_{CA}}^D{f_{BD}}^E - 2\,\tfrac{C(G)}{\dim G}\,\eta_{AB}\,\delta^E_C\Big) \\
&= -i\Big({f_{FE}}^C{f_{CB}}^D{f_{AD}}^E + {f_{FE}}^C{f_{CA}}^D{f_{BD}}^E\Big) \\
&= \mathrm{Tr}\Big(T^{AdjG}_F\{T^{AdjG}_B, T^{AdjG}_A\}\Big) \;=\; 0.
\end{aligned}$$

All information on $T^{AdjG}_A T^{AdjG}_B$ is in the structure constants and eq. (2.47) is essentially empty.

2.3.3 The cases of $U(N)$, $SU(N)$ and $U(1)$

The Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ calls for a more detailed discussion of these groups and Lie algebras. As explained in section 2.1, $U(N) \sim SU(N) \times U(1)$.

Firstly, $U(1)$ is the one-dimensional ($\dim G = 1$) abelian group of phase rotations. Structure constants vanish, $C(U(1)) = 0$, all its unitary representations have complex dimension one. In other words, $U(1)$ transformations of a set of fields can always be written in a complex basis where

$$\phi_i \qquad \longrightarrow \qquad e^{i\alpha\, Q_i}\, \phi_i. \tag{2.49}$$

The $U(1)$ charges Q_i are the eigenvalues of the single generator and α is the corresponding group parameter. The quadratic Dynkin index for a field with charge Q_i is

$$T(Q_i) = Q_i^2 \tag{2.50}$$

since representations can be simply labelled by the value of the generator.

As mentioned earlier, unitary groups $U(N)$ can be represented as $N \times N$ complex matrices U verifying $U^\dagger U = UU^\dagger = \mathbb{I}_N$, where $\mathbb{I}_N$ is the unit matrix in N complex dimensions. Since U has $2N^2$ real matrix elements submitted to N^2 conditions, the unitary group $U(N)$ has N^2 continuous real parameters. For $SU(N)$, the supplementary condition $\det U = 1$ (unimodularity) removes one parameter. Hence, the Lie algebra of $SU(N)$ has $N^2 - 1$ generators. Unitarity and unimodularity of group elements respectively imply hermiticity and tracelessness of the generators. Hence, a basis of the Lie algebra of $SU(N)$ in the fundamental representation is a basis of $N \times N$ complex hermitian traceless matrices. As mentioned in subsection 2.3.2, we use normalization

$$\mathrm{Tr}(T_A^N T_B^N) = \frac{1}{2}\delta_{AB}, \qquad\qquad T(N) = \frac{1}{2}. \tag{2.51}$$

The simplest case is $SU(2)$. A basis for 2×2 hermitian traceless matrices is given by Pauli matrices

$$T_1^2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T_2^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad T_3^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.52}$$

which provide a set of generators for the fundamental representation of $SU(2)$. With normalization (2.51), the $SU(2)$ Lie algebra is

$$[T_A, T_B] = i\epsilon_{ABC} T_C, \tag{2.53}$$

which is the algebra of spin operators in quantum mechanics. Generators for the adjoint representation have matrix elements ${(T_A)_B}^C = -i\,\epsilon_{ABC}$. The Casimir number of $SU(2)$ is then given by $(-i)^2\epsilon_{ADC}\epsilon_{BCD} = 2\delta_{AB} \rightarrow C(SU(2)) = 2$.

The theory of strong interactions, quantum chromodynamics (QCD), is based on the symmetry group $SU(3)$. A basis of generators for the fundamental three-dimensional complex Lie algebra representation with normalization $T(3) = 1/2$ is provided by the eight *Gell-Mann matrices*:[15]

$$\begin{aligned}
&\lambda_i = \tfrac{1}{2}\begin{pmatrix} \multicolumn{2}{c}{\sigma_i} & \begin{matrix}0\\0\end{matrix} \\ 0 & 0 & 0 \end{pmatrix}, && i = 1,2,3, \qquad (SU(2)\text{ subalgebra}),\\
&\lambda_4 = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0 \end{pmatrix}, && \lambda_5 = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0 \end{pmatrix},\\
&\lambda_6 = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0 \end{pmatrix}, && \lambda_7 = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0 \end{pmatrix},\\
&\lambda_8 = \tfrac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{2.54}$$

The conjugate representation $\overline{3}$ has generators of opposite signs. The adjoint representation has real dimension eight (it is denoted by "8"). Then:

$$3 \otimes \overline{3} = 1 \oplus 8 \qquad \longrightarrow \qquad 3T(\overline{3}) + 3T(3) = 6T(3) = 3 = T(1) + T(8) = 0 + C(G).$$

The Casimir number for the fundamental representation 3 is $C(3) = 8T(3)/3 = 4/3$.

Since $[\lambda_8, \lambda_i] = 0$, the four matrices $\{\lambda_i, \lambda_8\}$ generate the maximal subalgebra $SU(2) \times U(1)$. In addition,

$$\left[\lambda_i\,, \begin{pmatrix} 0 & \vec{A} \\ \vec{A}^\dagger & 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & \sigma_i\vec{A} \\ -\vec{A}^\dagger\sigma_i & 0 \end{pmatrix}, \qquad \left[\lambda_8\,, \begin{pmatrix} 0 & \vec{A} \\ \vec{A}^\dagger & 0 \end{pmatrix}\right] = \frac{\sqrt{3}}{2}\begin{pmatrix} 0 & \vec{A} \\ -\vec{A}^\dagger & 0 \end{pmatrix}$$

indicate that the two-component complex doublet $\vec{A}$ obtained by combining linearly $\lambda_4, \ldots, \lambda_7$ is a doublet (representation 2) of the $SU(2)$ subalgebra, with charge $\sqrt{3}/2$ under λ_8. In other words the embedding $SU(3) \supset SU(2) \times U(1)$ is defined by

$$\begin{aligned}
&3 = (2, \tfrac{1}{2\sqrt{3}}) + (1, -\tfrac{1}{\sqrt{3}}), \qquad \overline{3} = (2, -\tfrac{1}{2\sqrt{3}}) + (1, \tfrac{1}{\sqrt{3}}),\\
&3 \times \overline{3} = 8 + 1 = (3,0) + (1,0) + (1,0) + (2, \tfrac{\sqrt{3}}{2}) + (2, -\tfrac{\sqrt{3}}{2})\\
&\qquad \Longrightarrow \qquad 8 = (3,0) + (1,0) + (2, \tfrac{\sqrt{3}}{2}) + (2, -\tfrac{\sqrt{3}}{2}).
\end{aligned}$$

[15] σ_i are Pauli matrices.

The center of the group $SU(3)$ has three elements. In the fundamental representation,

$$\{\mathbb{I}_3\,,\, e^{2i\pi/3}\,\mathbb{I}_3\,,\, e^{4i\pi/3}\mathbb{I}_3\}. \tag{2.55}$$

It is the discrete group $\mathbb{Z}_3$. The $\mathbb{Z}_3$ transformation of an irreducible representation induces an additive *triality number* with value 0, 1 or -1 (modulo 3) attached to the representation. It is 1 for the fundamental 3, -1 for its conjugate $\overline{3}$ and then 0 for the invariant (singlet) 1 or for the adjoint 8. Quarks in representation 3 have then triality 1, antiquarks in $\overline{3}$ have triality -1, and since confinement in QCD only allows $SU(3)$–singlet bound states, hadrons have n_q quarks and $n_{\overline{q}}$ antiquarks with $n_q - n_{\overline{q}} = 0$ (mod 3). The *baryon number* is $B = (n_q - n_{\overline{q}})/3$. It is an integer for all $SU(3)$–singlet hadronic bound states.

Similar results hold for all $SU(N)$ groups. They have center $\mathbb{Z}_N$ with elements $\exp(2ik\pi/N)\,\mathbb{I}_N$ ($k = 0, 1, \ldots, N-1$) in the fundamental representation and each irreducible representation carries then an additive *N–ality number*.

Chapter 3

Gauge theories

The concept of gauge invariance is fundamental in the construction of the Standard Model of strong, weak and electromagnetic interactions of elementary particles. The gauge principle prescribes that, given the gauge symmetry group and the transformations of the fields, the quantum field theory is uniquely defined. This prescription imposes the existence of gauge bosons (photon, gluons, $W^\pm$ and Z^0), the form of their interactions with quarks and leptons, and also, since $W^\pm$ and Z^0 are massive spin 1 fields, the existence of a spin zero particle commonly named "Higgs boson". The prediction of this scalar particle is associated to the mechanism of spontaneous symmetry breaking used to give masses to $W^\pm$ and Z^0 (see chapter 4). Hence, all information in the construction of the theory is algebraic.

In chapter 1, as a simple example, we have considered quantum electrodynamics, in which the gauge symmetry is abelian. The gauge group $U(1)$ has one parameter and the theory includes one *gauge field* $A_\mu(x)$ describing the massless photon. The photon does not have any charge and only interacts with charged particles.

A field theory with a non-abelian gauge symmetry group is called a *Yang-Mills theory*, or simply a *gauge theory*. The transition from abelian to non-abelian gauge symmetry is not straightforward and the aim of this chapter is to obtain the most general gauge theory Lagrangian which can be quantized in perturbation theory. This is only possible if the gauge symmetry is a compact Lie group. Hence, in this chapter, generators are hermitian and the antisymmetric structure constants f_{ABC} [defined in eq. (2.32)] are equal to the "true" structure constants ${f_{AB}}^C$. We will only use f_{ABC} in all equations.

3.1 Kinetic Lagrangian

To formulate the principle of gauge invariance, we first consider a classical field theory describing real scalar fields $\varphi^i(x)$ and spinor fields $\psi^I(x)$ without masses and interactions. The kinetic Lagrangian only includes propagation terms depending on derivatives of the fields. The resulting field equations (Euler-Lagrange equations) should be the massless Klein-Gordon equation for scalars and Dirac equation for spinors. Since left-handed (ψ_L^I) and right-handed (ψ_R^I) Weyl spinors separately transform under the Lorentz group, they are independent fields in the massless theory:

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\varphi^i)(\partial^\mu\varphi^i) + i\overline{\psi}_{LI}\gamma^\mu\partial_\mu\psi_L^I + i\overline{\psi}_{RJ}\gamma^\mu\partial_\mu\psi_R^J. \tag{3.1}$$

The numbers of scalar, left-handed spinor and right-handed spinor fields will be denoted by N_s, N_L and N_R. In principle, N_L and N_R could be different. The field equations are

$$\Box\,\varphi^i = i\gamma^\mu\partial_\mu\psi_L^I = i\gamma^\mu\partial_\mu\psi_R^I = 0\,. \tag{3.2}$$

As observed earlier[1], right-handed Weyl spinors can be replaced by left-handed (charge conjugate) spinors and we may then choose to work with left-handed (or right-handed) spinors only:

$$(\ \psi_L^I\ ,\ \psi_R^I\ ) \quad\longrightarrow\quad (\ \psi_L^I\ ,\ \psi_L^{C\,I}\ ).$$

The kinetic Lagrangian density has a large continuous symmetry. Firstly, the kinetic term of scalar fields is invariant under transformations

$$\varphi^i \longrightarrow \varphi^{i\,\prime} = O^i_j\varphi^j, \qquad \sum_{k=1}^{N_s} O^k_i O^k_j = \delta_{ij}. \tag{3.3}$$

The $N_s \times N_s$ matrix O is real and orthogonal, $O^\tau O = \mathbb{I}$, the symmetry group is then $O(N_s)$. Then, working with left-handed spinors only, transformations

$$\psi_L^I \longrightarrow \psi_L^{I\,\prime} = U^I_J\psi_L^J, \qquad (U^\dagger)^I_J U^J_K = \delta^I_K \tag{3.4}$$

leave the Lagrangian (3.1) unchanged. These transformations belong to the $U(N_L)$ group of *chiral transformations*, acting on left-handed massless Weyl spinors.

To introduce gauge invariance, we postulate that a subgroup of this global[2] symmetry is turned into a local symmetry of the Lagrangian density. More precisely, we

[1]Paragraph 1.4.1, eq. (1.88) with $m = 0$.
[2]A global symmetry has parameters which do not depend on the space-time point.

require that transformations

$$\begin{aligned}
\varphi^j &\longrightarrow \varphi^{j\,\prime} = \left(e^{i\alpha^A T^s_A}\right)^j_{\ k} \varphi^k, \\
\psi^J_L &\longrightarrow \psi^{J\,\prime}_L = \left(e^{i\alpha^A T^\ell_A}\right)^J_{\ K} \psi^K_L, \\
\psi^J_R &\longrightarrow \psi^{J\,\prime}_R = \left(e^{i\alpha^A T^r_A}\right)^J_{\ K} \psi^K_R,
\end{aligned} \tag{3.5}$$

with parameters α^A depending on the space-time point

$$\alpha^A \longrightarrow \alpha^A(x),$$

are *gauge symmetries* of the theory. The elements of the gauge group have being defined from its Lie algebra, with generators T^s_A, T^ℓ_A and T^r_A for respectively the representations of real scalar fields, left-handed and right-handed spinors. The (compact) Lie algebra is

$$[T^\sharp_A, T^\sharp_B] = i f_{ABC} T^\sharp_C, \qquad \sharp = s, \ell \text{ ou } r. \tag{3.6}$$

Since scalar fields are real, generators T^s_A are imaginary and antisymmetric.

3.2 Gauge fields and covariant derivatives

Since

$$\begin{aligned}
(\partial_\mu \varphi^j)' &= \left(e^{i\alpha^A T^s_A}\right)^j_{\ k} \partial_\mu \varphi^k + \left[\partial_\mu \left(e^{i\alpha^A T^s_A}\right)^j_{\ k}\right] \varphi^k, \\
(\partial_\mu \psi^J_L)' &= \left(e^{i\alpha^A T^\ell_A}\right)^J_{\ K} \partial_\mu \psi^K_L + \left[\partial_\mu \left(e^{i\alpha^A T^\ell_A}\right)^J_{\ K}\right] \psi^K_L, \\
(\partial_\mu \psi^J_R)' &= \left(e^{i\alpha^A T^r_A}\right)^J_{\ K} \partial_\mu \psi^K_R + \left[\partial_\mu \left(e^{i\alpha^A T^r_A}\right)^J_{\ K}\right] \psi^K_R,
\end{aligned} \tag{3.7}$$

the kinetic Lagrangian (3.1) is not gauge invariant: to restore the symmetry, the second terms need to be compensated. This is achieved by introducing one *gauge field*[3] $A^A_\mu(x)$ for each independent transformation and then one gauge field for each generator (hence the index A). Transformations of these gauge fields are then derived from the requirement that *covariant derivatives*

$$\begin{aligned}
D_\mu \varphi^j &= \partial_\mu \varphi^j - i A^A_\mu (T^s_A)^j_{\ k} \varphi^k, \\
D_\mu \psi^J_L &= \partial_\mu \psi^J_L - i A^A_\mu (T^\ell_A)^J_{\ K} \psi^K_L, \\
D_\mu \psi^J_R &= \partial_\mu \psi^J_R - i A^A_\mu (T^r_A)^J_{\ K} \psi^K_R,
\end{aligned} \tag{3.8}$$

[3] Also named *gauge potentials*, by analogy with the four-vector of potentials in Maxwell theory.

transform in the same way[4] as the fields themselves. We then need

$$\begin{array}{rclcl} D_\mu\varphi^j & \longrightarrow & D_\mu\varphi^{j\,\prime} & = & \left(e^{i\alpha^A T^s_A}\right)^j_{\ k} D_\mu\varphi^k, \\ D_\mu\psi^J_L & \longrightarrow & D_\mu\psi^{J\,\prime}_L & = & \left(e^{i\alpha^A T^\ell_A}\right)^J_{\ K} D_\mu\psi^K_L, \\ D_\mu\psi^J_R & \longrightarrow & D_\mu\psi^{J\,\prime}_R & = & \left(e^{i\alpha^A T^r_A}\right)^J_{\ K} D_\mu\psi^K_R. \end{array} \tag{3.9}$$

We first consider infinitesimal transformations. For scalar fields,

$$\begin{array}{rcl} \delta\varphi^i & = & i\alpha^A (T^s_A)^i_j \varphi^j, \\ \delta\partial_\mu\varphi^i & = & i\alpha^A (T^s_A)^i_j (\partial_\mu\varphi^j) + i(\partial_\mu\alpha^A)(T^s_A)^i_j\varphi^j, \\ \delta D_\mu\varphi^i & = & i\alpha^A (T^s_A)^i_j D_\mu\varphi^j \\ & & +i\varphi^j\{(\partial_\mu\alpha^A)(T^s_A)^i_j - (\delta A^A_\mu)(T^s_A)^i_j - iA^A_\mu\alpha^B([T^s_A,T^s_B])^i_j\}. \end{array}$$

To cancel the last line, we need

$$\sum_A (\delta A^A_\mu) T^s_A = \sum_A \left(\partial_\mu\alpha^A T^s_A + A^A_\mu \sum_{B,C} f_{ABC}\,\alpha^B T^s_C\right).$$

Since generators are normalised by

$$\mathrm{Tr}(T^s_A T^s_B) = T(s)\,\delta_{AB}, \tag{3.10}$$

we obtain

$$\begin{array}{rcl} A^A_\mu & \longrightarrow & A^A_\mu + \delta A^A_\mu, \\ \delta A^A_\mu & = & \partial_\mu\alpha^A + \sum_{B,C} f_{ABC}\,A^B_\mu\alpha^C. \end{array} \tag{3.11}$$

Of course, this expression does not depend on the representation of scalar fields. We would have obtained the same transformation of gauge fields by considering left-handed or right-handed fermions.

The same argument applied to the general, non infinitesimal, case of transformations (3.5) leads to

$$\sum_A A^{A\,\prime}_\mu T^\sharp_A = e^{i\alpha^B T^\sharp_B}[i\partial_\mu + \sum_A A^A_\mu T^\sharp_A] e^{-i\alpha^C T^\sharp_C}, \qquad \sharp = s, \ell \text{ or } r, \tag{3.12}$$

in matrix notation (with a sum on indices B and C). Again, eq. (3.12) imposes the same transformation of gauge fields for $T_A = T^s_A$, T^ℓ_A or T^r_A. It reduces to (3.11) if parameters α^A are infinitesimal and to first order in these α_A.

[4]Hence the terminology "covariant".

According to transformations (3.9), kinetic terms of the Lagrangian density are gauge invariant if derivatives ∂_μ are replaced by the appropriate covariant derivative D_μ. We then find that

$$\mathcal{L}_0 = \frac{1}{2}(D_\mu\varphi^i)(D^\mu\varphi^i) + \overline{\psi}_{LJ} i\gamma^\mu D_\mu \psi_L^J + \overline{\psi}_{RJ} i\gamma^\mu D_\mu \psi_R^J \tag{3.13}$$

is gauge invariant and contains *scalar–gauge fields* and *fermion–gauge fields* interactions. This Lagrangian does not however include propagation terms for the gauge fields: it does not depend on derivatives $\partial_\mu A_\nu^A$. The next step is then to construct the gauge-invariant kinetic Lagrangian density of gauge fields.

3.3 Gauge curvature, gauge kinetic terms

To construct gauge kinetic terms, we first introduce the *gauge curvature*[5]

$$F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + \sum_{BC} f_{ABC}\, A_\mu^B A_\nu^C. \tag{3.14}$$

Since the structure constants f_{ABC} are antisymmetric, $F_{\mu\nu}^A = -F_{\nu\mu}^A$, its infinitesimal gauge transformation is

$$F_{\mu\nu}^{A\,\prime} = F_{\mu\nu}^A + \sum_{B,C} f_{ABC}\, F_{\mu\nu}^B \alpha^C. \tag{3.15}$$

To derive this variation, Jacobi identity

$$\sum_B (f_{ACB} f_{BDE} + f_{ADB} f_{BEC} + f_{AEB} f_{BCD}) = 0,$$

is used. The quantity

$$-\frac{1}{4} F_{\mu\nu}^A F^{A\,\mu\nu}, \tag{3.16}$$

is then gauge invariant since $F_{\mu\nu}^A \delta F^{\mu\nu\,A} = f_{ABC} F_{\mu\nu}^A F^{\mu\nu\,B}\alpha^C = 0$. It is the *Yang-Mills Lagrangian* describing the propagation and interactions of non-abelian gauge fields.

A matrix notation is often useful. Take an arbitrary set of generators $\{T_A\}$ and define

$$A_\mu = A_\mu^A\, T_A, \qquad F_{\mu\nu} = F_{\mu\nu}^A\, T_A, \qquad \alpha = \alpha^A\, T_A. \tag{3.17}$$

The matrix of gauge curvatures is then

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu, A_\nu] \tag{3.18}$$

[5] Also named *field strength*.

and gauge variations are

$$\begin{aligned} \delta A_\mu &= \partial_\mu \alpha + i\,[\alpha, A_\mu], \\ \delta F_{\mu\nu} &= i\,[\alpha, F_{\mu\nu}]. \end{aligned} \tag{3.19}$$

According to eq. (2.28), gauge curvatures transform in the adjoint representation of the gauge Lie algebra. In addition, the variation of gauge potentials includes a specific nonlinear term $\partial_\mu \alpha$ which also exists in the abelian case.

3.4 Gauge coupling constants

Interactions involving gauge fields included in (3.16) and (3.13) have a strength characterized by *gauge coupling constants*. In general, the gauge group G has the stucture of a product $G = G_1 \times G_2 \times \ldots = \prod_a G_a$. Each factor G_a is either simple or $U(1)$. For instance, the gauge group of the Standard Model is $SU(3) \times SU(2) \times U(1)$. Two generators T_A and T_B taken in two different factors G_a commute and $f_{ABC} = 0$ for all generators T_C in G. Suppose that we substitute

$$A^A_\mu \qquad \longrightarrow \qquad g^A A^A_\mu, \qquad\qquad (g^A : \text{nonzero real numbers})$$

(no sum on A) in covariant derivatives and in gauge curvatures. And also

$$F^A_{\mu\nu} \;\longrightarrow\; g^A \left[\partial_\mu A^A_\nu - \partial_\nu A^A_\mu + \sum_{BC} (g^A)^{-1} g^B g^C f_{ABC} A^B_\mu A^C_\nu \right] \equiv g^A F^A_{\mu\nu}$$

(no sum on A). In order that gauge transformation (3.15) stays true, we need that

$$g^A = g^B = g^C \qquad \Longleftrightarrow \qquad f_{ABC} \neq 0. \tag{3.20}$$

This condition implies that gauge invariance only allows one coupling constant g_a for each factor G_a in the gauge group. The number of parameters included in gauge field interactions in (3.16) and (3.13) is then equal to the number of simple or $U(1)$ factors in the gauge group.

3.5 Gauge-invariant kinetic Lagrangian

To summarize, if the gauge group is $G = \prod_a G_a$, the part of the gauge-invariant Lagrangian density which depends on derivatives of the fields is given by

$$\mathcal{L}_{kin.} = -\frac{1}{4} F^A_{\mu\nu} F^{A\,\mu\nu} + \frac{1}{2} (D_\mu \varphi^i)(D^\mu \varphi^i) + \overline{\psi}_{LJ} i\gamma^\mu D_\mu \psi^J_L + \overline{\psi}_{RJ} i\gamma^\mu D_\mu \psi^J_R. \tag{3.21}$$

It includes all terms necessary for the propagation of spin 0, 1/2 and 1 fields (or more precisely of helicities 0, $\pm 1/2$ and ± 1), as well as all interaction terms involving gauge fields. Covariant derivatives are

$$\begin{aligned} D_\mu \varphi^j &= \partial_\mu \varphi^j - i \sum_A g^A A_\mu^A (T_A^s)^j_k \varphi^k, \\ D_\mu \psi_L^J &= \partial_\mu \psi_L^J - i \sum_A g^A A_\mu^A (T_A^\ell)^J_K \psi_L^K, \\ D_\mu \psi_R^J &= \partial_\mu \psi_R^J - i \sum_A g^A A_\mu^A (T_A^r)^J_K \psi_R^K, \end{aligned} \tag{3.22}$$

and gauge curvatures read

$$F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + g^A \sum_{BC} f_{ABC} A_\mu^B A_\nu^C. \tag{3.23}$$

Gauge coupling constants g^A verify (3.20). Finally, infinitesimal transformations are given by (3.5) and by

$$\begin{aligned} \delta A_\mu^A &= (g^A)^{-1} \partial_\mu \alpha^A + \sum_{B,C} f_{ABC}\, A_\mu^B\, \alpha^C, \\ \delta F_{\mu\nu}^A &= \sum_{B,C} f_{ABC}\, F_{\mu\nu}^B\, \alpha^C. \end{aligned} \tag{3.24}$$

Gauge coupling constants are dimensionless numbers. In fact, since action $S = \int d^4x\, \mathcal{L}$ is dimensionless, $\mathcal{L}$ has dimension four (in energy units) and it then follows from $\mathcal{L}_{kin.}$ that the so-called *canonical dimensions* of fields φ^i, A_μ^A and $\psi_{L,R}^I$ are respectively 1, 1 and 3/2.

The coherence rules of the quantum field theory allow to add to this kinetic Lagrangian various terms without derivatives ∂_μ of the fields. They should not depend on gauge fields A_μ^A. The rules for these terms can be stated as follows:

- All terms should be gauge invariant.[6]
- All terms should have dimension four.
- The previous condition should not be achieved using parameters with (strictly) *negative* dimension.

Violating these rules leads to a non-renormalizable theory: the classical Lagrangian cannot be quantized, it does not admit any sensible perturbation theory.

[6] And of course Lorentz invariant.

Allowed terms fall in two categories. Firstly, mass terms which are quadratic in scalar and fermion fields. They contribute to the Euler-Lagrange equations by terms linear in the fields. Secondly, terms cubic or quartic in the fields contribute to field equations by nonlinear terms and describe interactions of scalar and fermion fields.

The conditions imposed by gauge invariance are simply obtained using the infinitesimal transformations

$$\begin{aligned}
\delta\varphi^j &= i\alpha^A (T^s_A)^j_k \varphi^k, \\
\delta\psi^J_L &= i\alpha^A (T^\ell_A)^J_K \psi^K_L, & \delta\overline{\psi}_{LJ} &= -i\alpha^A \overline{\psi}_{LK} (T^\ell_A)^K_J, \\
\delta\psi^J_R &= i\alpha^A (T^r_A)^J_K \psi^K_R, & \delta\overline{\psi}_{RJ} &= -i\alpha^A \overline{\psi}_{RK} (T^r_A)^K_J.
\end{aligned} \tag{3.25}$$

3.6 Mass terms

These contributions to the Lagrangian are quadratic in scalar and spinor fields:

$$\begin{aligned}
\mathcal{L}_{m.} &= \mathcal{L}_{m.s.} + \mathcal{L}_{m.f.}, \\
\mathcal{L}_{m.s.} &= -\tfrac{1}{2}(m^2)^{ij}\varphi^i\varphi^j, \\
\mathcal{L}_{m.f.} &= -(M)^I_J \overline{\psi}_{LI}\psi^J_R - (M^\dagger)^I_J \overline{\psi}_{RI}\psi^J_L.
\end{aligned} \tag{3.26}$$

The matrix m^2 is real and symmetric. Its eigenvalues are the (masses)^2 of the scalar fields. Invariance under infinitesimal gauge transformations (3.25) gives the conditions

$$\begin{aligned}
(m^2)^{kj}(T^s_A)^i_k + (m^2)^{ik}(T^s_A)^j_k = 0, \\
(M)^I_J (T^r_A)^J_K - (T^\ell_A)^I_J (M)^J_K = 0
\end{aligned} \tag{3.27}$$

on the mass matrices. In matrix notation,

$$T^s_A(m^2) + (m^2)T^{s\,\tau}_A = [T^s_A, m^2] = 0, \qquad M\,T^r_A - T^\ell_A\,M = 0, \tag{3.28}$$

since T^s_A is antisymmetric and imaginary. Hence,

- Masses of scalar fields are constant in each irreducible representation of the gauge group.
- Nonzero fermion masses require the existence of a multiplet of left-handed *and* right-handed fermions in the same representation, in other words, Dirac spinors $\psi^I = \psi^I_L + \psi^I_R$ in this representation are required. The exception would be a Majorana spinor in a real representation of the gauge group.[7]

[7] Reality of the representation implies that the particle and the antiparticle have the same transformations.

A mass term for gauge fields, which would take the form

$$\frac{1}{2}\mathcal{M}^2_{AB}A^A_\mu A^{\mu\,B},$$

is clearly forbidden by invariance under gauge transformations (3.24). To each gauge symmetry corresponds then a massless gauge boson.

3.7 Yukawa interactions

The most general *fermion–scalar interaction* is of the form

$$\begin{aligned} \mathcal{L}_{Yuk.} &= \lambda_{iJ}^{\;K}\varphi^i(\overline{\psi}_{LK}\psi^J_R) + (\lambda_{iJ}^{\;K})^*\varphi^i(\overline{\psi}_{RJ}\psi^K_L) \\ &= \tfrac{1}{2}A_{iJ}^{\;K}\varphi^i(\overline{\psi}_K\psi^J) + \tfrac{1}{2}B_{iJ}^{\;K}\varphi^i(\overline{\psi}_K\gamma_5\psi^J), \end{aligned} \tag{3.29}$$

with $A_{iJ}^{\;K} = \lambda_{iJ}^{\;K} + (\lambda_{iK}^{\;J})^*$ and $B_{iJ}^{\;K} = -\lambda_{iJ}^{\;K} + (\lambda_{iK}^{\;J})^*$. Since $\overline{\psi}_K\psi^J$ is hermitian, the "scalar" couplings are hermitian:

$$(A_{iJ}^{\;K})^* = A_{iK}^{\;J}.$$

In contrast, "pseudoscalar" couplings verify

$$(B_{iJ}^{\;K})^* = -B_{iK}^{\;J},$$

a consequence of the antihermitian property of $\overline{\psi}_K\gamma_5\psi^J$. Gauge invariance requires

$$\lambda_{jJ}^{\;K}(T^s_A)^j_i - \lambda_{iJ}^{\;M}(T^\ell_A)^K_M + \lambda_{iM}^{\;K}(T^r_A)^M_J = 0. \tag{3.30}$$

The canonical dimension of $\varphi^i(\overline{\psi}_{LK}\psi^J_R)$ is four and Yukawa couplings $\lambda_{iJ}^{\;K}$ are then dimensionless numbers.

3.8 Scalar interactions

Cubic or quartic *scalar interaction* terms are

$$\Delta_s(\varphi^i) = -\frac{1}{3}\alpha_{ijk}\,\varphi^i\varphi^j\varphi^k - \frac{1}{4}\beta_{ijkl}\,\varphi^i\varphi^j\varphi^k\varphi^l. \tag{3.31}$$

The coupling constants α_{ijk} and β_{ijkl} are real and symmetric under permutations of their indices. They are constrained by gauge invariance, which requires

$$\frac{\partial}{\partial\varphi^i}\Delta_s(\varphi^j)\delta\varphi^i = 0 \qquad \longrightarrow \qquad \frac{\partial}{\partial\varphi^i}\Delta_s(\varphi^j)(T^s_A)^i_k\varphi^k = 0$$

for all values of the fields and of A, *i.e.*

$$\begin{aligned} 0 &= \alpha_{ljk}(T^s_A)^l_i + \alpha_{ilk}(T^s_A)^l_j + \alpha_{ijl}(T^s_A)^l_k, \\ 0 &= \beta_{mjkl}(T^s_A)^m_i + \beta_{imkl}(T^s_A)^m_j + \beta_{ijml}(T^s_A)^m_k + \beta_{ijkm}(T^s_A)^m_l, \end{aligned} \tag{3.32}$$

for all values of i, j, k, l. While coupling constants β_{ijkl} are dimensionless, parameters α_{ijk} have dimension one (in energy units).

Usually, all scalar terms without derivative are collected in the scalar potential [8]

$$V(\varphi^i) = \frac{1}{2}(m^2)_{ij}\varphi^i\varphi^j + \frac{1}{3}\alpha_{ijk}\varphi^i\varphi^j\varphi^k + \frac{1}{4}\beta_{ijkl}\varphi^i\varphi^j\varphi^k\varphi^l. \tag{3.33}$$

The scalar potential plays a central role in spontaneous symmetry breaking[9].

3.9 The complete Lagrangian density

We can now write the most general Lagrangian density describing scalar fields (spin or helicity 0), spinor fields (spin 1/2 or helicity $\pm 1/2$) and vector fields (helicity ± 1) admissible in the framework of quantum field theory. It is the sum of the gauge invariant kinetic Lagrangian (3.21), mass terms (3.26), Yukawa interactions (3.29) and scalar interactions (3.31):

$$\mathcal{L} = \mathcal{L}_{kin.} + \mathcal{L}_{m.f.} + \mathcal{L}_{Yuk.} - V(\varphi^i). \tag{3.34}$$

This theory is completely determined by:

1. The gauge group (the local symmetry group), which also defines the structure constants f_{ABC} and the number of gauge coupling constants g^A.

2. The representation of scalar fields φ^i (*i.e.* the choice of scalar generators T^s_A).

3. The representation of spinor fields ψ^I_L and ψ^J_R (*i.e.* the choices of generators T^ℓ_A and T^r_A).

The choices of the gauge group and of the scalar representation are free. Consistency of the quantum field theory requires that the representation of the spinor fields is such that the anomaly coefficient [eq. (2.41)] vanishes. Explicitly, *anomaly cancellation* requires

$$\begin{aligned} &d^\ell_{ABC} = d^r_{ABC}\,, \\ &d^\ell_{ABC} = T(\ell)^{-1}\,\mathrm{Tr}(T^\ell_A\{T^\ell_B, T^\ell_C\}), \qquad d^r_{ABC} = T(r)^{-1}\,\mathrm{Tr}(T^r_A\{T^r_B, T^r_C\}). \end{aligned} \tag{3.35}$$

[8] Notice that the Lagrangian density includes then $-V(\varphi^i)$.

[9] See chapter 4.

This condition is in particular trivially verified if the representations of left-handed and right-handed spinor fields are identical ($T^\ell_A = T^r_A$). In this case, the gauge theory can be formulated in terms of Dirac fermions, it is free of anomalies and gauge interactions respect parity invariance. This is the case of quantum chromodynamics (QCD, strong interactions) and of QED. But anomaly-cancellation also allows different representations ($T^\ell_A \neq T^r_A$): this is the case in the Standard Model of strong and electroweak interactions. Gauge interactions may then violate parity, as do weak interactions.

At this stage, scalar and fermion masses, if allowed by gauge invariance [*i.e.* if allowed by conditions (3.27)] are free parameters. Similarly, Yukawa couplings and scalar interactions are free parameters. Gauge symmetry predicts that gauge fields are massless, but we will see in the next chapter that this theory can display spontaneous symmetry breaking in its scalar sector, generating masses for some of the gauge fields. In the Standard Model of strong and electroweak interactions, this mechanism is at the origin of the masses of gauge bosons $W^\pm$ and Z^0.

Chapter 4

Spontaneous symmetry breaking

A quantum field theory of spin one fields only exists if each spin one field is associated with a gauge symmetry. Gauge invariance implies then that the field is massless, describing two states with helicities ± 1. This agrees with the needs of quantum electrodynamics (a massless photon) and of quantum chromodynamics (eight massless gluons mediating strong interactions). It is however not appropriate for weak interactions which require massive fields describing $W^{\pm}$ and Z^0 spin one bosons.

The description of Z^0 and $W^{\pm}$ interactions in the framework of quantum field theory calls for a generalization of gauge theories admitting massive vector fields. This generalization is based on the phenomenon of *spontaneous symmetry breaking*, which uses two fundamental results, *Goldstone theorem*, which applies to continuous global or local symmetries of a field theory, and the *Higgs mechanism* which occurs when gauge symmetries are spontaneously broken. In addition to the generation of massive spin one fields, spontaneous breaking of gauge symmetries also predicts the appearance of elementary particles with zero spin called *Higgs bosons*.

4.1 Goldstone theorem

A free real scalar field is a solution of Klein-Gordon equation $(\Box + m^2)\varphi = 0$, which follows from the Lagrangian

$$\mathcal{L}_{free} = \frac{1}{2}(\partial_\mu \varphi)(\partial_\mu \varphi) - \frac{m^2}{2}\varphi^2. \tag{4.1}$$

It can be written as the wave packet[1]

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(k)e^{-ikx} + a^*(k)e^{ikx}\right], \tag{4.2}$$

[1]The integration measure $d^3k/(2\omega_k)$ is Lorentz invariant since the four vector k is 'on-shell", $k^2 = m^2$.

with wave vector $k^\mu = (\omega_k, \vec{k})$, $\omega_k = \sqrt{m^2 + \vec{k}^2}$. If the mass m^2 vanishes, the scalar field admits a constant zero mode, the contribution to expansion (4.2) of $\vec{k} = 0$. This is an arbitrary number which is the *vacuum expectation value (v.e.v.)* of the field,[2]

$$\langle \varphi(x) \rangle = v. \tag{4.3}$$

If the mass is not zero, there is a gap: $\omega_k^2 \geq m^2 > 0$ and $\langle \varphi(x) \rangle = 0$. An interacting scalar has Lagrangian[3]

$$\mathcal{L}_{scal.} = \frac{1}{2}(\partial_\mu \varphi)(\partial_\mu \varphi) - V(\varphi), \tag{4.4}$$

the field equation is

$$\Box \varphi = -\frac{\partial V}{\partial \varphi}, \tag{4.5}$$

and a zero mode solution $\varphi(x) = \langle \varphi(x) \rangle = v$ corresponds to the minimum of the scalar potential[4]

$$\left. \frac{\partial V}{\partial \varphi} \right|_{\varphi = v} = 0. \tag{4.6}$$

For a free, massless field, $V \equiv 0$ and v is arbitrary. Hence, the choice of the parameters in the potential decides if the scalar field has a zero or nonzero vacuum value.

Since the field φ is a Lorentz scalar, its expectation value v does not break relativistic invariance. It is then an implicit parameter of the field theory allowed by special relativity. It does not play any role for a free massless field (since the Lagrangian only depends on $\partial_\mu \varphi$ in this case), but it has physical implications in the interacting theory. In particular, since scalar fields in general transform under continuous symmetries of the Lagrangian, a nonzero expectation value v breaks in general some of these symmetries.

Let us then consider a set of N_s real scalar fields $\varphi^i(x)$, $i = 1, \ldots, N_s$. The Lagrangian density of the field theory is

$$\mathcal{L}_{scal.} = \frac{1}{2}(\partial_\mu \varphi^i)(\partial^\mu \varphi^i) - V(\varphi^i), \tag{4.7}$$

the scalar potential $V(\varphi^i)$ being a polynomial of fourth order (or less) in the scalar fields. It can be seen as a general gauge field theory with gauge group G in the limit where all spinor and vector fields vanish, a limit consistent with the field equations and the symmetries of the theory. The kinetic term of theory (4.7) is invariant under global

[2] A constant φ is solution of the Klein-Gordon equation if $m^2 = 0$.

[3] All other fields in the theory are assumed zero. We nevertheless use a partial derivative with respect to φ since other fields are in general present.

[4] Classically, the minimum of the scalar potential determines the ground state of the theory. In the quantum field theory, the ground state is called the *vacuum state* and quantum corrections apply to the potential and to its ground state.

$O(N_s)$ rotations and we assume that the potential is invariant under transformations of a global symmetry group $G_{scal.} \subset O(N_s)$. Small variations

$$\delta\varphi^i = i\alpha^A \, (T^s_A)^i{}_j \, \varphi^j \, , \tag{4.8}$$

where matrices T^s_A are generators of the Lie algebra of $G_{scal.}$ in the representation of the scalar fields[5], leave then V invariant:

$$\delta V(\varphi^i) = i\alpha^A \frac{\partial V}{\partial \varphi^i} (T^s_A)^i{}_j \varphi^j = 0. \tag{4.9}$$

Since the Lie group G of the underlying gauge theory is certainly a symmetry of the scalar Lagrangian, $G \subset G_{scal.}$, but $G_{scal.}$ could be in principle larger than G.

Field (Euler-Lagrange) equations of theory (4.7),

$$\Box\varphi^i + \frac{\partial V}{\partial \varphi^i} = 0, \tag{4.10}$$

admit constant solutions $\varphi^i(x) = c^i$ if

$$\left[\frac{\partial V}{\partial \varphi^i} \right]_{\varphi^i = c^i} = 0. \tag{4.11}$$

Classically, these constant solutions are stable under small perturbations of the scalar fields if the potential has a local minimum: $V(c^i + \delta\varphi^i) \geq V(c^i)$ for arbitrary small variations $\delta\varphi^i$. It is the case if the matrix of second partial derivatives

$$\left[\frac{\partial^2 V}{\partial \varphi^i \partial \varphi^j} \right]_{\varphi^k = c^k}$$

has only positive or zero eigenvalues. In the quantum field theory, the global minimum of the potential only is a stable state, the *vacuum state* for which $\langle \varphi^i(x) \rangle = v^i$. The vacuum state must exist and the scalar potential $V(\varphi^i)$ must then be bounded below. Since the potential is a fourth-order polynomial, this is in general a condition on quartic terms in V.

Eq. (4.9) indicates that for all values φ^i of the scalar fields φ^i,

$$\frac{\partial V}{\partial \varphi^i} \, (T^s_A)^i{}_j \varphi^j = 0, \qquad\qquad A = 1, 2, \ldots \tag{4.12}$$

for all generators of the Lie algebra of $G_{scal.}$. The derivative with respect to φ^j is

$$\frac{\partial^2 V}{\partial \varphi^i \partial \varphi^j} \, (T^s_A)^i{}_k \varphi^k + \frac{\partial V}{\partial \varphi^i} \, (T^s_A)^i{}_j = 0, \qquad\qquad \forall A, \quad \forall j. \tag{4.13}$$

[5] We consider a compact symmetry G_s and generators are then hermitian, $T^s_A = {T^s_A}^\dagger$ and imaginary (since scalar fields are real).

Then, at the vacuum state $\varphi^i = v^i$,

$$\left(\frac{\partial V}{\partial \varphi^j}\right)_{\varphi^i = v^i} = 0, \qquad \left(\frac{\partial^2 V}{\partial \varphi^i \partial \varphi^j}\right)_{\varphi^i = v^i} (T^s_A)^i{}_k v^k = 0, \qquad \forall A, \quad \forall j. \tag{4.14}$$

If the vacuum state of the theory is nontrivial, $\langle \varphi^i \rangle = v^i \neq 0$, we can divide the generators of the symmetry algebra in two categories denoted by $\tilde{T}^s_A$ and $\not{T}^s_A$ and defined by

$$\begin{array}{lrcl} i: & (\tilde{T}^s_A)^i{}_j \, v^j & = & 0, \\ ii: & (\not{T}^s_A)^i{}_j \, v^j & \neq & 0. \end{array} \tag{4.15}$$

The vacuum state is left invariant by the symmetries $\tilde{T}^s_A$, which generate the algebra of a subgroup H of $G_{scal.}$.[6] Generators $\not{T}^A_s$ correspond to symmetries of $G_{scal.}$ *spontaneously broken* at the vacuum state. They are symmetries of the Lagrangian and of the field equations which are not respected by the vacuum state. The spontaneous breaking can be characterized by the breaking pattern

$$G \qquad \longrightarrow \qquad H, \qquad\qquad\qquad H \subset G.$$

The invariance of the scalar Lagrangian under the transformations of $G_{scal.}$,

$$\varphi^i \rightarrow \varphi^{i'} = U^i{}_j \, \varphi^j, \tag{4.16}$$

where $U = \exp(i\alpha^A T^s_A)$ implies that if v^i is a vacuum state of the theory, then

$$v^{i'} = U^i{}_j \, v^j \tag{4.17}$$

is also a vacuum state for all $U \in G_{scal.}$: if some symmetry is spontaneously broken (and then in general $v^i \neq v^{i'}$) the vacuum state is continuously degenerate. If U_H is an element of H, $(U_H)^i{}_j v^j = v^i$, and then

$$(U U_H U^{-1})^i{}_j v'^j = v'^i. \tag{4.18}$$

For each U, the vacuum expectation values $v^{i'}$ are left invariant by $U U_H U^{-1}$ and the little group of $v^{i'}$ is then H again: all degenerate vacua have the same little group H.

Notice that if the scalar fields include one or several fields which are invariant under $G_{scal.}$, these directions can have nonzero vacuum value without breaking any symmetry.

A theory in which the vacuum state is $\langle \varphi^i \rangle = v^i$ can always be reformulated in terms of fields

$$\tilde{\varphi}^i = \varphi^i - v^i, \tag{4.19}$$

[6] H is called the *little group* or *stabilizer* of the vacuum expectation value v^i.

for which the vaccum state is $\langle\tilde{\varphi}^i\rangle = 0$. The Lagrangian density is

$$\mathcal{L}_{scal.} = \frac{1}{2}(\partial_\mu\tilde{\varphi}^i)(\partial^\mu\tilde{\varphi}^i) - \tilde{V}(\tilde{\varphi}^i), \qquad \tilde{V}(\tilde{\varphi}^i) = V(\tilde{\varphi}^i + v^i). \tag{4.20}$$

The potential also reads

$$\begin{aligned}\tilde{V}(\tilde{\varphi}^i) &= \langle V\rangle + \left[\frac{\partial V}{\partial\varphi^i}\right]_{\varphi^k=v^k}\tilde{\varphi}^i + \frac{1}{2}\left[\frac{\partial^2 V}{\partial\varphi^i\partial\varphi^j}\right]_{\varphi^k=v^k}\tilde{\varphi}^i\,\tilde{\varphi}^j \\ &\quad + \text{cubic and quartic terms,}\end{aligned} \tag{4.21}$$

where $\langle V\rangle = V(v^i)$. The first term, the constant $\langle V\rangle$, is without physical significance: we can always choose the energy of the vacuum state to be zero. The second term linear in $\tilde{\varphi}^i$ vanishes by eq. (4.14). The quadratic term includes the matrix of the (squared) masses of the new scalar fields:

$$(\mathcal{M}^2_{\tilde{\varphi}})_{ij} = \left[\frac{\partial^2 V}{\partial\varphi^i\partial\varphi^j}\right]_{\varphi^k=v^k}. \tag{4.22}$$

Since v^i corresponds to the global minimum of V, this mass matrix has only positive or zero eigenvalues.

The second eq. (4.14) indicates however that the scalar fields

$$(T^s_A)^i{}_j\, v^j \tag{4.23}$$

are massless. There is one such massless scalar for each spontaneously broken symmetry.

We have then obtained *Goldstone theorem* [10, 11]:

> To each continuous (global or local) symmetry of the action which is not a symmetry of the vacuum state corresponds a massless (real) scalar field, the *Goldstone boson* of the spontaneously broken symmetry.

The explicit form of the potential, as function of the scalar fields φ^i, has not been used in the derivation of Goldstone theorem. In a quantum field theory, the quantum-corrected potential function is in general different from the classical potential appearing in the Lagrangian (which is a fourth order polynomial). As a result, the unbroken symmetry group H (the little group of the vacuum state) of the quantum theory could be different from the classical little group. But in any case, to each broken symmetry corresponds an exactly massless spin zero state.

The simplest example of a spontaneously broken continuous symmetry uses a complex scalar field ϕ with scalar potential

$$V(\phi,\phi^\dagger) = \frac{\lambda}{2}\Big(\phi^\dagger\phi - \frac{\mu^2}{2\lambda}\Big)^2 = -\frac{\mu^2}{2}\phi^\dagger\phi + \frac{\lambda}{2}(\phi^\dagger\phi)^2 + \frac{\mu^4}{8\lambda}. \tag{4.24}$$

The real constant λ is positive in order to have a bounded potential and we choose $\mu^2 > 0$. The potential is invariant under phase rotations of ϕ:

$$\phi \qquad \longrightarrow \qquad \phi' = e^{i\alpha Q}\phi, \tag{4.25}$$

The invariance group is then $G_{scal.} = U(1)$. An infinitesimal variation is $\delta\phi = i\alpha Q\phi$, the number Q being the single generator of the Lie algebra and α the arbitrary parameter. The minimum of the potential is reached if $\langle\phi^\dagger\phi\rangle = \mu^2/2\lambda$, or if

$$\langle\phi\rangle = e^{i\beta}\, v, \qquad v = \sqrt{\mu^2/2\lambda}, \qquad \beta \text{ real}.$$

The action of $G_{scal.} = U(1)$ changes the value of β, which encodes the degeneracy of the vacuum state. Symmetry $U(1)$ is then spontaneousy broken. Next, we introduce the new field $\tilde\phi$ for which the minimum of the potential corresponds to $\langle\tilde\phi\rangle = 0$,

$$\tilde\phi = \phi - e^{i\beta}v,$$

and we split $\tilde\phi$ in two real fields:

$$\tilde\phi(x) = \frac{1}{\sqrt{2}}e^{i\beta}\,[A(x) + iB(x)].$$

Finally, the action of $U(1)$ symmetry (4.25) indicates that we may freely choose $\beta = 0$. In terms of the fields A and B, the potential writes:

$$V(A,B) = \frac{1}{2}\mu^2A^2 + \frac{\lambda}{\sqrt{2}}v\,A(A^2+B^2) + \frac{\lambda}{8}(A^2+B^2)^2, \qquad v = \sqrt{\mu^2/2\lambda}. \tag{4.26}$$

It describes a real field A with mass μ interacting with a massless real field B, the Goldstone boson of the spontaneousy broken $U(1)$ symmetry.

4.2 Spontaneous breaking of gauge symmetries

According to Goldstone theorem, spontaneous breaking of continuous symmetries predicts the existence of massless spin zero bosons, which are not seen in Nature. It turns out that the spontaneous breaking of local continuous symmetries (gauge symmetries) predicts instead that the associated gauge fields acquire masses. The mechanism provides then a description of massive spin one particles. The Goldstone bosons are not "independent" particles: they are the third components with helicity zero of the spin one states. In a particular gauge (called *unitary gauge*), the Goldstone bosons are actually absorbed by the massive gauge fields. Even if this phenomenon is commonly known under the name "Higgs mechanism", it has been first described in three articles by Brout and Englert [12], Higgs [13] and Guralnik, Hagen and Kibble [14][7]. This section describes in general terms the spontaneous breaking of gauge symmetries. It is then applied to massive $W^\pm$ and Z^0 bosons in the Standard Model, in the example presented in next section.

To describe the Higgs mechanism, we need (real) scalar fields φ^i, the gauge bosons A_μ^A of the gauge symmetry and their Lagrangian. Spinor fields do not play any role and are omitted. According to the results of previous chapter, the field theory is

$$\mathcal{L} = \mathcal{L}_{gauge} + \frac{1}{2}(D_\mu\varphi^i)(D^\mu\varphi^i) - V(\varphi^i), \tag{4.27}$$

with

$$\mathcal{L}_{gauge} = -\frac{1}{4}F_{\mu\nu}^A F^{A\,\mu\nu}, \qquad F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + g^A \sum_{BC} f_{ABC} A_\mu^B A_\nu^C,$$

and[8]

$$D_\mu\varphi^i = \partial_\mu\varphi^i - i\sum_A g^A A_\mu^A (T_A^s)^{ij}\varphi^j. \tag{4.28}$$

Suppose as before that the potential leads to the vacuum state $\langle\varphi^i\rangle = v^i \neq 0$. Introducing new shifted fields with zero vacuum expectation values $\tilde\varphi^i = \varphi^i - v^i$, the Lagrangian density becomes

$$\begin{aligned}\mathcal{L} \;=\; & \mathcal{L}_{gauge} + \frac{1}{2}(D_\mu\tilde\varphi^i)(D^\mu\tilde\varphi^i) - V(\tilde\varphi^i + v^i) \\ & +\frac{1}{2}\sum_{A,B} g^A g^B [v^i (T_A^s T_B^s)^{ij} v^j] A_\mu^A A^{B\,\mu} - i\sum_A g^A A_\mu^A (D^\mu\tilde\varphi^i)(T_A^s)^{ij} v^j,\end{aligned} \tag{4.29}$$

since $D_\mu v^i = -i\sum_A g^A A_\mu^A (T_A^s)^{ij} v^j$. Two new contributions quadratic in the fields are produced: firstly, gauge boson mass terms of the form

$$\frac{1}{2}\mathcal{M}_1^{AB} A_\mu^A A^{B\,\mu}, \qquad \mathcal{M}_1^{AB} = g^A g^B [v^i (T_A^s T_B^s)^{ij} v^j]. \tag{4.30}$$

Secondly, propagation terms mixing gauge fields A_μ^A and scalar fields:

$$-i\sum_A g^A A_\mu^A (\partial^\mu\tilde\varphi^i)(T_A^s)^{ij} v^j = i\sum_A g^A (\partial^\mu A_\mu^A)\tilde\varphi^i (T_A^s)^{ij} v^j + \partial^\mu(\ldots). \tag{4.31}$$

Clearly, only gauge fields associated to broken symmetries for which $(T_A^s)^{ij} v^j \neq 0$ contribute to expressions (4.30) and (4.31).

[7] The article by Brout and Englert was submitted and published first. It is cited in Refs. [13, 14]. Some prototype incomplete versions appeared earlier, for instance in a first article by Higgs [15].

[8] Scalar fields are real. Generators T_A^s are then imaginary and antisymmetric.

Theory (4.27) is invariant under gauge transformations

$$\begin{aligned}
\varphi^i \qquad &\longrightarrow \qquad \varphi'^i = [e^{i\alpha^A T^s_A}]^{ij}\varphi^j, \\
\sum_A g^A A^A_\mu T^s_A \qquad &\longrightarrow \qquad \sum_A g^A A'^A_\mu T^s_A = e^{i\alpha^B T^s_B}\left[i\partial_\mu + \sum_A g^A A^A_\mu T^s_A\right] e^{-i\alpha^C T^s_C}.
\end{aligned} \tag{4.32}$$

However, decomposition $\varphi^i = \tilde{\varphi}^i + v^i$ and the Lagrangian density (4.29), which depend on the vacuum v^i, are not explicitly invariant: since the original theory (4.27) is gauge-invariant, we can use any gauge to define the shifted scalar fields $\tilde{\varphi}^i$ and express the theory in terms of these new fields.

To identify the physical content of the theory and also eliminate the mixing term (4.31), it is convenient to use another parameterization of the scalar fields. We have N real scalar fields φ^i, we assume that the Lie algebra of the gauge group G has dimension M and that the vacuum v^i breaks p symmetries. There is then p Goldstone bosons and, necessarily, $p < N$. The p vectors $(\not{T}^s_A)^{ij}v^j$, $A = 1, \ldots, p$, generate the Goldstone bosons in the N-dimensional space of the scalar fields. We may then represent Goldstone bosons by

$$[\exp{(i\xi^A(x)\not{T}^s_A)}]^{ij}v^j,$$

with p fields $\xi^A(x)$. The $N - p$ other fields are then included in expression

$$\varphi^i = [\exp{(i\xi^A(x)\not{T}^s_A)}]^{ij}\left(h^j(x) + v^j\right), \tag{4.33}$$

where the vector $h^i(x)$ contains $N-p$ real fields, orthogonal to the directions of Goldstone bosons,

$$h^i(x)(\not{T}^s_A)^{ij}v^j = 0. \tag{4.34}$$

Note that the vector $h(x)v^i$ is always a particular solution of this equation since generators are antisymmetric. In addition, since the generators $\tilde{T}^s_A$ of the little group H (unbroken gauge symmetries) leave the vacuum v^i invariant, $(\tilde{T}^s_A)^{ij}v^j = 0$, they also leave $h(x)v^i$ invariant: there exists always at least one real field in $h^i(x)$ (and then $N-p > 0$) and this field is invariant under the little group H of unbroken symmetries.

Expression (4.33) can be directly introduced in the gauge-invariant Lagrangian density (4.27). But it can also be simplified using a gauge transformation (4.32) with parameters

$$\alpha^A(x) = -\xi^A(x) \tag{4.35}$$

whenever A corresponds to a broken symmetry, $T^s_A = \not{T}^s_A$, and $\alpha^A = 0$ for generators of the little group H of v^i. Then,

$$\varphi^i \qquad \longrightarrow \qquad \varphi'^i = [\exp{(-i\xi^A\not{T}^s_A)}]^{ij}\varphi^j = h^i(x) + v^i.$$

Expressed in terms of φ'^i, the Lagrangian density does not depend on Goldstone bosons which were eliminated by the gauge choice:

$$\begin{aligned}\mathcal{L}_{unit.} &= -\frac{1}{4}F^A_{\mu\nu}F^{A\,\mu\nu} + \frac{1}{2}\mathcal{M}_1^{AB}A^A_\mu A^{B\,\mu} \\ &\quad +\frac{1}{2}(D_\mu h^i)(D^\mu h^i) - V(h^i + v^i),\end{aligned} \tag{4.36}$$

the gauge boson mass matrix being

$$\mathcal{M}_1^{AB} = g^A g^B [v^i (\mathcal{T}^s_A \mathcal{T}^s_B)^{ij} v^j], \tag{4.37}$$

as in (4.30). The mixing term is absent due to equation (4.34). This choice corresponds to the *unitary gauge*. The theory describes then a set of p massive spin one vector fields corresponding to the spontaneously broken gauge symmetries, interacting with the massless gauge bosons of the little group H and with a multiplet of scalar *Higgs bosons* h^i which includes at least one field. These scalar fields h^i transform in some representation of H. The structure of all interactions is completely determined by the original gauge theory and by the vacuum state v^i. Gauge interactions are fixed by the structure constants and the generators of G and gauge boson masses are related (by the diagonalization of the mass matrix $\mathcal{M}_1^{AB}$) to the values of $\langle\varphi^i\rangle = v^i$, and then to the parameters of the scalar potential which determines the vacuum state.

Another derivation of the unitary gauge amounts to eliminate the mixing term (4.31) by a gauge transformation of the gauge fields associated to the broken symmetries. The Lagrangian density (4.29) contains in particular the following quadratic terms[9]:

$$\begin{aligned}X &\equiv \frac{1}{2}\left[(\partial_\mu\tilde{\varphi}^i)(\partial^\mu\tilde{\varphi}^i) + \mathcal{M}_1^{AB}A^A_\mu A^{B\,\mu} - 2ig^A A^A_\mu(\partial^\mu\tilde{\varphi}^i)(\mathcal{T}^s_A)^{ij}v^j\right] \\ &= -\frac{1}{2}\left[g^A A^A_\mu(\mathcal{T}^s_A)^{ij}v^j + i(\partial_\mu\tilde{\varphi})^i\right]\left[g^B A^{B\,\mu}(\mathcal{T}^s_B)^{ik}v^k + i(\partial^\mu\tilde{\varphi})^i\right].\end{aligned}$$

Compare the last expression with gauge transformation (4.32) multiplied by v^j, to first order since we only consider here quadratic terms of the Lagrangian:

$$g^A A'^A_\mu(\mathcal{T}^s_A)^{ij}v^j = g^A A^A_\mu(\mathcal{T}^s_A)^{ij}v^j + (\partial_\mu\alpha^A)(\mathcal{T}^s_A)^{ij}v^j.$$

We then choose parameters such that $\alpha^A(\mathcal{T}^s_A)^{ij}v^j - i\tilde{\varphi}^i = -ih^i$, where the fields h^i represent the $N-p$ components of $\tilde{\varphi}^i$ left unchanged by the gauge transformation:

$$\tilde{\varphi}^i = h^i - i\alpha^A(\mathcal{T}^s_A)^{ij}v^j, \qquad h^i(\mathcal{T}^s_A)^{ij}v^j = 0.$$

[9]Genrators are antisymmetric.

These equations correspond to the unitary gauge (4.35) applied to the parameterization (4.33) of fields to first order, and to the orthogonality condition (4.34). We then arrive at

$$\begin{aligned} X &= -\frac{1}{2}\left[g^A A'^A_\mu (\mathcal{T}^s_A)^{ij} v^j\right]\left[g^B A'^{B\mu}(\mathcal{T}^s_B)^{ik} v^k\right] + \frac{1}{2}(\partial_\mu h^i)(\partial^\mu h^i) \\ &= \frac{1}{2}\mathcal{M}_1^{AB} A'^A_\mu A'^{B\mu} + \frac{1}{2}(\partial_\mu h^i)(\partial^\mu h^i). \end{aligned}$$

The unitary gauge is particularly useful to the identification of the physical states of the theory. It is however at the origin of important complications if it would be adopted to quantify the theory and develop perturbation theory. More convenient, less intuitive gauges are actually chosen to compute quantum probabilities.

Group theory provides a simple necessary condition if one wishes to break a certain gauge group G into a subgroup H. The gauge fields of the theory always transform in the adjoint representation $AdjG$ of G. Scalar fields used to induce spontaneous symmetry breaking $G \to H$ transform in some representation R_s, in general reducible, of G. But $H \subset G$ and any representation of G decomposes as a sum of representations of H. In particular, one can write

$$AdjG = AdjH \oplus R_H, \qquad R_H = R_{H,1} \oplus \ldots \oplus R_{H,k}.$$

Gauge bosons made massive by the Higgs mechanism transform in representation R_H of H and the Goldstone bosons absorbed in the massive spin one fields should then transform in the same representation R_H. Then, in order to break G into H using scalar fields in representation R_s, it is necessary that[10]:

1. Representation R_s contains a direction invariant under H: this is the direction of the vacuum state $\varphi^i = v^i$.

2. Representation R_s, when decomposed in representations of H, includes the representation R_H of the massive gauge fields and of the Goldstone bosons.

In other words, the decomposition of R_s in representations of H should be of the form

$$R_S = 1 \oplus R_H \oplus R_{others}\,,$$

where 1 is the direction of the vacuum v^i. The Higgs bosons transform in representation $1 \oplus R_{others}$ of H (and R_{others} may be absent, as in the following examples). This is however not sufficient: we still have to find a scalar potential such that its global minimum is precisely in the direction v^i, invariant under H.

[10]These necessary conditions do not make sense if the little group H is trivial, in which case all gauge symmetries are spontaneously broken.

As a first example, consider again the spontaneous breaking of symmetry $U(1)$, as in previous section, but now with gauge invariance. As before, the scalar potential is

$$V = \frac{\lambda}{2}\left(\phi^\dagger\phi - \frac{\mu^2}{2\lambda}\right)^2,$$

leading to $\langle\phi\rangle = e^{i\beta}v$, $v = \sqrt{\mu^2/2\lambda}$. Instead of writing $\phi = \frac{1}{\sqrt{2}}e^{i\beta}[A + iB + v]$, we use the parameterization

$$\phi(x) = e^{i\sigma(x)+i\beta}\left[\frac{1}{\sqrt{2}}h(x) + v\right],$$

with two real scalar fields σ and h. Since the scalar potential only depends on $\phi^\dagger\phi$, σ is massless: it is the Goldstone boson. We then change the gauge,

$$\phi(x) \qquad \longrightarrow \qquad \phi'(x) = e^{-i\sigma(x)-i\beta}\phi(x) = \frac{1}{\sqrt{2}}h(x) + v,$$

to go to the unitary gauge which only contains the Higgs boson h, with mass μ. And the gauge boson of symmetry $U(1)$ acquires a mass $M = gQv$, according to gauge transformation (4.25).

4.3 An example: the complex scalar doublet

This example corresponds to the scalar sector of the Standard Model. It is used to give masses to three gauge fields, W^+, W^- and Z^0. Hence, we need to spontaneously break three gauge symmetries and produce three Goldstone bosons. Since there is in addition at least one Higgs boson, we need at least four real scalar fields. We then consider the minimal case, which uses a doublet of complex scalar fields

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \qquad\qquad H^\dagger = (\, H_1^\dagger \quad H_2^\dagger \,), \tag{4.38}$$

with scalar potential

$$V(H, H^\dagger) = -\mu^2(H^\dagger H) + \frac{\lambda}{2}(H^\dagger H)^2. \tag{4.39}$$

In this expression,

$$\mu^2 > 0, \qquad \lambda > 0, \qquad \mu^2,\ \lambda \ \text{ real},$$

and $H^\dagger H = H_1^\dagger H_1 + H_2^\dagger H_2$. The minimum of V is at

$$\langle H^\dagger H\rangle = \mu^2/\lambda. \tag{4.40}$$

The potential (4.39) is invariant under the transformations of the group[11] $SU(2) \times U(1)_Y$ [which is equivalent to $U(2)$]. These symmetries act on doublet H with

$$\begin{array}{lrcl} U(1)_Y: & H \longrightarrow H' & = & e^{i\alpha Y} H, \\ SU(2): & H \longrightarrow H' & = & UH, \end{array} \tag{4.41}$$

where the (2×2) matrix U is unitary, $U^\dagger U = \mathbb{I}_2$, and unimodular ($\det U = 1$). We can write

$$U = e^{iw^a T_a}, \qquad T_a = \frac{1}{2}\sigma_a, \qquad a = 1,2,3, \tag{4.42}$$

matrices σ_a denoting the three Pauli matrices. Generators of the $SU(2)$ Lie algebra verify

$$[T_a, T_b] = i\epsilon_{abc} T_c, \qquad \epsilon_{abc} = -\epsilon_{bac} = \epsilon_{cab}, \qquad \epsilon_{123} = 1.$$

The real constant Y is the generator of $U(1)_Y$. since its value is arbitrary, we choose

$$Y = -\frac{1}{2} \tag{4.43}$$

and scalars are then in representation $(2, -1/2)$ of $SU(2) \times U(1)_Y$.

In fact, the potential (4.39) has a symmetry larger than $SU(2) \times U(1)_Y$: if we use real fields $\varphi_1 = \operatorname{Re} H_1$, $\varphi_2 = \operatorname{Im} H_1$, $\varphi_3 = \operatorname{Re} H_2$, $\varphi_4 = \operatorname{Im} H_2$, then $H^\dagger H = \sum_{i=1}^4 \varphi_i^2$ which is invariant under rotations $O(4) \sim SU(2) \times SU(2)$ of the four real fields. We will however only consider the subgroup $SU(2) \times U(1)_Y$, which will be promoted to a gauge symmetry, as in the Standard Model.

At this stage, $SU(2) \times U(1)_Y$ transformation parameters α and w^a can be constant or local. The minimum condition (4.40) is of course invariant under the full symmetry group and the vacuum state is continuously degenerate. One can then choose

$$\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix}, \qquad v = \sqrt{\frac{2\mu^2}{\lambda}}, \tag{4.44}$$

and all degenerate vacuum states are then obtained by acting on $\langle H \rangle$ with an $SU(2) \times U(1)_Y$ transformation (4.41) with constant parameters.

The vacuum state (4.44) spontaneously breaks $SU(2) \times U(1)_Y$. One easily verifies that the residual symmetry leaving $\langle H \rangle$ invariant is

$$\exp\,(iw[T_3 + Y])\,\langle H \rangle = \langle H \rangle, \qquad T_3 + Y = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.45}$$

The little group of $\langle H \rangle$ is then $U(1)_Q$, generated by

$$Q = T_3 + Y. \tag{4.46}$$

[11] Index Y identifies the group $U(1)_Y$ in contrast with another $U(1)$ group which appears below.

The action of Q on the components of H is

$$U(1)_Q: \qquad H_1 \longrightarrow H_1, \qquad\qquad H_2 \longrightarrow e^{-iw}H_2.$$

Three symmetries are then spontaneously broken anf the symmetry breaking pattern is

$$SU(2)\times U(1)_Y \qquad \longrightarrow \qquad U(1)_Q. \tag{4.47}$$

The four real components of H split in three Goldstone bosons and one massive state, the *Higgs boson*.

We next couple the complex scalar doublet to $SU(2)\times U(1)_Y$ gauge fields. In so doing, $SU(2)\times U(1)_Y$ becomes a local, gauge symmetry. We must introduce gauge fields W_μ^a, $a=1,2,3$, for $SU(2)$, and B_μ for $U(1)_Y$ and also two gauge coupling constants g and g' for respectively $SU(2)$ and $U(1)_Y$. The gauge-invariant Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}W^a_{\mu\nu}W^{a\,\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + (D_\mu H)^\dagger(D^\mu H) - V(H,H^\dagger), \tag{4.48}$$

and, using the results of section 3.2,

$$\begin{aligned} W^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon_{abc}W^b_\mu W^c_\nu, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\ D_\mu H &= \partial_\mu H - \tfrac{i}{2}gW^a_\mu\sigma_a H - ig'B_\mu Y H, \qquad (Y=-\tfrac{1}{2}). \end{aligned} \tag{4.49}$$

The form of the covariant derivative of H follows from gauge transformation (4.41). With vacuum expectation value $\langle H\rangle$, the particle content of the theory is most easily identified using the unitary gauge which, according to the previous section, is obtained with the parameterization

$$\exp\left[i\xi^1(x)T_1 + i\xi^2(x)T_2 + i\xi(x)(T_3-Y)\right]\langle H\rangle,$$

of the Goldstone fields, the three broken symmetries being generated by T_1, T_2 and

$$T_3 - Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We then write

$$H = \frac{1}{\sqrt{2}}\exp\left[i\xi^1(x)T_1 + i\xi^2(x)T_2 + i\xi(x)(T_3-Y)\right]\begin{pmatrix} h(x)+v \\ 0 \end{pmatrix}, \tag{4.50}$$

with a real scalar field $h(x)$ which will be the single Higgs boson of the theory. Verifying orthogonality condition (4.34) requires care since we use here complex fields and non-antisymmetric generators. In our case, the condition becomes

$$h_i(x)(T^s_A)^{ij}\langle H^*_j\rangle = [h_i(x)(T^s_A)^{ij}\langle H^*_j\rangle]^\dagger \qquad\qquad h_1(x)=h(x), \qquad h_2(x)=0.$$

And it is actually verified:

$$\langle H\rangle^\tau T_1 \begin{pmatrix} h(x) \\ 0 \end{pmatrix} = \langle H\rangle^\tau T_2 \begin{pmatrix} h(x) \\ 0 \end{pmatrix} = 0,$$

while

$$\langle H\rangle^\tau [T_3 - Y] \begin{pmatrix} h(x) \\ 0 \end{pmatrix} = vh(x)$$

and $h(x)$ is a real field. It follows from (4.50) that the complex doublet reduces to

$$H_{unit.} = \frac{1}{\sqrt{2}} \begin{pmatrix} h(x) + v \\ 0 \end{pmatrix} \tag{4.51}$$

in unitary gauge. The Lagrangian density in this gauge is obtained by simply replacing H by $H_{unit.}$ in the gauge invariant expression (4.48). In order to diagonalize mass terms of gauge fields, we use the following redefinitions:

$$\begin{array}{rclrclcl} W^+_\mu &=& \frac{1}{\sqrt{2}}(W^1_\mu - iW^2_\mu), & W^-_\mu &=& \frac{1}{\sqrt{2}}(W^1_\mu + iW^2_\mu) &=& (W^+_\mu)^\dagger, \\ Z_\mu &=& \cos\theta_W W^3_\mu - \sin\theta_W B_\mu, & A_\mu &=& \sin\theta_W W^3_\mu + \cos\theta_W B_\mu. && \end{array} \tag{4.52}$$

The mixing angle θ_W is defined by

$$\cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}}, \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}. \tag{4.53}$$

In the Standard Model, θ_W is *Weinberg angle*, or the *weak mixing angle*. With definitions

$$\begin{array}{rclrcl} A_{\mu\nu} &=& \partial_\mu A_\nu - \partial_\nu A_\mu, & Z_{\mu\nu} &=& \partial_\mu Z_\nu - \partial_\nu Z_\mu, \\ W^+_{\mu\nu} &=& \partial_\mu W^+_\nu - \partial_\nu W^+_\mu, & W^-_{\mu\nu} &=& \partial_\mu W^-_\nu - \partial_\nu W^-_\mu, \end{array}$$

we then obtain:

$$\begin{aligned} \mathcal{L}_{unit.} = & -\frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{2}W^+_{\mu\nu}W^{-\,\mu\nu} \\ & +\frac{1}{2}\frac{g^2+g'^2}{4}v^2 Z_\mu Z^\mu + \frac{g^2}{4}v^2\, W^+_\mu W^{-\,\mu} \\ & +\frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \mu^2 h^2 - \sqrt{\frac{\lambda}{2}}\mu\, h^3 - \frac{\lambda}{8}h^4 + \frac{\mu^4}{2\lambda} \\ & +\mathcal{L}_{int.}. \end{aligned} \tag{4.54}$$

The first line includes propagation terms for gauge fields, the second the masses of the gauge bosons associated with broken symmetries and the third is the Lagrangian of the Higgs boson field. Finally, $\mathcal{L}_{int.}$ includes the gauge interactions:

$$\begin{aligned}\mathcal{L}_{int.} &= -ig\sin\theta_W\Big[(A^\mu + \mathrm{cotg}\theta_W\, Z^\mu)(W^+_{\mu\nu}W^{-\,\nu} - W^-_{\mu\nu}W^{+\,\nu}) \\ &\qquad -(A_{\mu\nu} + \mathrm{cotg}\theta_W\, Z_{\mu\nu})W^{+\,\mu}W^{-\,\nu}\Big] \\ &\quad +(g\sin\theta_W)^2\Big[(A_\mu W^{+\,\mu})(A_\nu W^{-\,\nu}) - (A_\mu A^\mu)(W^+_\nu W^{-\,\nu}) \\ &\qquad +\mathrm{cotg}\theta_W\Big\{(Z_\mu W^{+\,\mu})(A_\nu W^{-\,\nu}) + (Z_\mu W^{-\,\mu})(A_\nu W^{+\,\nu}) \\ &\qquad\qquad -2(A_\mu Z^\mu)(W^+_\nu W^{-\,\nu})\Big\} \\ &\qquad +\mathrm{cotg}^2\theta_W\Big\{(Z_\mu W^{+\,\mu})(Z_\nu W^{-\,\nu}) - (Z_\mu Z^\mu)(W^+_\nu W^{-\,\nu})\Big\}\Big] \\ &\quad -\frac{1}{2}g^2\left[(W^+_\mu W^{-\,\mu})^2 - (W^+_\mu W^{+\,\mu})(W^-_\nu W^{-\,\nu})\right] \\ &\quad +\frac{g^2}{4}(h^2+2vh)\left[W^+_\mu W^{-\,\mu} + \frac{1}{2\cos^2\theta_W}Z_\mu Z^\mu\right].\end{aligned} \tag{4.55}$$

The theory describes then:

- A massless gauge boson[12] A_μ associated with the unbroken symmetry $U(1)_Q$;
- A complex field and its conjugate, W^-_μ and W^+_μ, with spin one and mass

$$M_W = \frac{gv}{2}. \tag{4.56}$$

  Under the exact (unbroken) gauge symmetry $U(1)_Q$, their charge is[13] $Q = \pm 1$;

- A field Z_μ with spin one and mass

$$M_Z = \frac{1}{2}\sqrt{g^2+g'^2}\,v = \frac{M_W}{\cos\theta_W} \geq M_W, \tag{4.57}$$

  invariant (neutral, without charge) under $U(1)_Q$;

- A real scalar field $h(x)$ invariant (neutral, without charge) under $U(1)_Q$ and with mass

$$m_h^2 = 2\mu^2 = \lambda v^2 = \frac{4\lambda}{g^2}M_W^2, \tag{4.58}$$

  the Higgs boson of the theory.

[12]The photon field in the Standard Model, where Q is the generator of the electric charge.
[13]Since $U(1)_Y$ does not act on the gauge fields of $SU(2)$, $Y=0$ and $Q=T^3$ for these states.

In expression (4.55), all gauge-boson interactions involving Z_μ are controlled by the coupling constant $g\cos\theta_W$. The field A_μ is the gauge field of the residual, unbroken symmetry $U(1)_Q$. Its gauge-boson inteactions are controlled by the coupling constant

$$g\sin\theta_W = \frac{gg'}{\sqrt{g^2+g'^2}}\,.$$

In fact, all interactions of A_μ arise from contributions to gauge curvatures $W^a_{\mu\nu}$ of combination

$$gW^3_\mu T_3 = g\sin\theta_W(A_\mu + \mathrm{cotg}\theta_\mathrm{W} Z_\mu)T_3 = g\sin\theta_W(A_\mu + \mathrm{cotg}\theta_\mathrm{W} Z_\mu)Q,$$

since $Y = 0$ for $SU(2)$ gauge fields. The covariant derivative acting on $H_{unit.}$ does not depend on A_μ: the Higgs boson parallel to the direction of the vacuum state $\langle H\rangle$, which exists in all theories with spontaneously broken symmetries, is necessarily neutral under the whole unbroken gauge group which leaves $\langle H\rangle$ invariant. In our example, there is a single Higgs boson.

The Lagrangian density defined by expressions (4.54) and (4.55) decribes in the unitary gauge the contributions of the bosonic fields of the Standard Model. It will then reappear in the next chapter, supplemented by the contributions of quarks and leptons.

Part II

The Standard Model

Chapter 5

Symmetries and Lagrangian

The Standard Model of Glashow [16], Salam [17] and Weinberg [18] describes strong, weak and electromagnetic interactions of quarks and leptons. Its Lagrangian follows the rules stated in chapter 3 and its precise structure is motivated by the observation of many elementary physical processes. Almost all its predictions have been verified with high to extremely high precision. The goal of this chapter is to construct the Lagrangian density of the Standard Model. A discussion of some of its phenomenological implications can be found in chapter **??**.

According to the gauge principle, choosing the gauge group and the gauge transformations of the spinor fields and of the scalar fields is sufficient to write the Lagrangian density of the gauge theory, and then the dynamical equations for all fields of the theory. The choice of the gauge group and some general properties of spinor fields representations are suggested by phenomenological (deduced from experiments) considerations:

1. As explained in section 1.4, electromagnetic intractions correspond to gauge invariance under phase rotations of spinor fields with electric charge eQ. The electromagnetic gauge symmetry is then $U(1)_Q$ ("Q" stands for the electric charge), its gauge boson (the photon field) is massless and the $U(1)_Q$ symmetry is exact (*i.e.* it is not spontaneously broken, which would give a mass to the photon). The photon does not have a charge and it only interacts with charged particles (the electromagnetic gauge group is abelian).

2. The theory of strong interactions of quarks is quantum chromodynamics (QCD). Quarks exist in three *colours* on which act colour transformations. These transformations are elements of the three-dimensional (fundamental) representation of the colour gauge symmetry group $SU(3)$. Since $SU(3)$ is eight-dimensional, there are eight gauge bosons called *gluons*. Since the QCD gauge group is non-abelian,

gluons have a colour charge and self-interactions.

3. Both electromagnetic and strong interactions conserve parity symmetry. Hence left-handed and right-handed charged and coloured spinor fields have identical $SU(3) \times U(1)_Q$ transformations.

4. Weak interactions are more subtle. For instance, beta decay of the neutron, $n \to p^+ e^- \overline{\nu}_e$, is a *charged current weak interaction.* At the level of quarks, the process is
$$d \quad \longrightarrow \quad u + e^- + \overline{\nu}_e.$$
It is described by the exchange of a (virtual) spin-one boson W^-, from the conversion of quark d in u, and producing the leptonic pair $e^- + \overline{\nu}_e$. Hence, charged-current weak interactions "connect" particles with electric charges differing by one unit (of the fundamental electric charge e). The associated gauge bosons $W^\pm$ have then electric charge $\pm e$: this implies that the gauge theory description of weak and electromagnetic interactions cannot be dissociated, hence the name *electroweak interactions.* The electroweak gauge group is then $SU(2)_L \times U(1)_Y$ and $U(1)_Q$ is a subgroup of $SU(2)_L \times U(1)_Y$.

5. Since any $U(1)_Q$ subgroup of $SU(2)_L \times U(1)_Y$ is necessarily of the form $Q = \alpha T_3 + Y$ with eigenvalues $\pm 1, 0$ for T_3 in the adjoint representation of $SU(2)_L$, the unit charge $Q = \pm 1$ of $W^\pm$ indicates that
$$Q = T_3 + Y. \tag{5.1}$$
Gauge bosons have $Y = 0$ (abelian gauge fields have zero $U(1)$ charge) and the four gauge bosons in $SU(2)_L \times U(1)_Y$ have then electric charges $Q = 1, -1, 0, 0$. The four gauge bosons are then identified with the photon γ, the two bosons $W^\pm$ mediating charged-current weak interactions and the electrically-neutral boson Z^0 which mediates *neutral-current weak interactions.* The terminology *weak isospin* and *weak hypercharge* is used to indicate $SU(2)_L$ and $U(1)_Y$ respectively. The values of the weak hypercharge Y will be non trivial for spinor and scalar fields: they are chosen to match fermion and scalar electric charges.

6. The electroweak gauge group $SU(2)_L \times U(1)_Y$ is spontaneously broken into $U(1)_Q$, as in section 4.3, producing the masses of $W^\pm$ and Z^0. Hence, the Standard model includes at least a complex $SU(2)_L$ doublet of scalar fields, and it predicts the existence of a massive spin zero particle, the Higgs boson.

7. Since electroweak gauge bosons do not have strong interactions, and since gluons do not have electroweak interactions, the gauge group of the Standard Model is

a product:

$$G_{SM} = SU(3) \times SU(2)_L \times U(1)_Y . \tag{5.2}$$

8. Charged-current weak interactions maximally break parity symmetry: gauge bosons $W^{\pm}$ only interact with left-handed quark and lepton Weyl spinors. Hence, left- and right-handed Weyl spinors have different transformation properties under $SU(2)_L \times U(1)_Y$ and all quark and lepton mass terms (with an exception in the neutrino sector) are then forbidden by gauge invariance. However, spontaneous symmetry breaking of G_{SM} into $SU(3) \times U(1)_Q$ implies, since the residual unbroken symmetry corresponds to parity-conserving interactions, that quark and lepton masses can be generated by the Higgs mechanism, in parallel with weak gauge boson masses.

In contrast with electromagnetic interactions where electric charge conservation can be directly observed, colour conservation can only be indirectly deduced from hadron physics. Non-abelian gauge theories have two related properties: firstly, they are weak in processes in which exchanged energies are large with respect to a certain characteristic scale Λ. This is called *asymptotic freedom*, a property which is experimentally verified in hard scattering processes. Secondly, the non-abelian force is strong in processes with energies smaller than or similar to Λ. In particular, bound states are always neutral under the non-abelian force. This is confinement. Then, all hadrons are "colourless" and isolated quarks or coloured bound states do not exist. Hence, colour conservation is not a straightforward feature in hadronic processes. Historically, quarks were introduced to explain the hadronic spectrum. In terms of quark bound states, mesons are $q\overline{q}$ states, baryons are qqq states and antibaryons $\overline{qqq}$ states. In particular, to reconcile hadronic bound states with quantum mechanics of identical fermions, a further quantum number of quarks, colour, was postulated to fully antisymmetrize the quantum state of three identical quarks. The existence of mesons $q\overline{q}$ (and the absence of qq or $\overline{qq}$ bound states) suggests a $U(N_c)$ or $SU(N_c)$ colour symmetry (N_c is the number of colours), while the existence of baryons made of three quarks points at $SU(3)$ ($N_c = 3$). Once $SU(3)$ is then used as a gauge group, in the framework of QCD, various hard scattering processes were used to confirm the postulate and to measure the scale Λ.

5.1 Gauge group and gauge bosons

The gauge principle associates a gauge field with each local symmetry and then with each generator of the gauge group. With gauge group $SU(3) \times SU(2)_L \times U(1)_Y$, the twelve gauge fields of the Standard Model are defined as:

$$\begin{array}{llrcll}
SU(3): & A^A_\mu, & A & = & 1,\ldots,8, & \text{(gluons)}, \\
SU(2)_L: & W^a_\mu, & a & = & 1,2,3, & \\
U(1)_Y: & B_\mu. & & & &
\end{array} \tag{5.3}$$

As already mentioned, the four gauge fields W^a_μ, $a = 1,2,3$ and B_μ will describe, after spontaneous symmetry breaking, electroweak gauge bosons W^+, W^-, Z^0 and the photon. According to eq. (3.14), the gauge curvatures appearing in the kinetic Lagrangian of the gauge fields are

$$\begin{array}{rrcl}
SU(3): & F^A_{\mu\nu} & = & \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + g_s\, f_{ABC} A^B_\mu A^C_\nu, \\
SU(2)_L: & W^a_{\mu\nu} & = & \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\,\epsilon_{abc}\, W^b_\mu W^c_\nu, \\
U(1)_Y: & B_{\mu\nu} & = & \partial_\mu B_\nu - \partial_\nu B_\mu.
\end{array} \tag{5.4}$$

The structure constants of $SU(3)$ and $SU(2)_L$ are respectively denoted by f_{ABC} and ϵ_{abc}. Under $G_{SM} = SU(3) \times SU(2)_L \times U(1)_Y$, gauge fields transform in the following representations:[1]

Gauge bosons	*Representation*	*Adjoint repres. of*	*Electric charges*
A^A_μ	$(\mathbf{8},\mathbf{1},0)$	$SU(3)$	0
W^a_μ	$(\mathbf{1},\mathbf{3},0)$	$SU(2)_L$	$+1,-1,0$
B_μ	$(\mathbf{1},\mathbf{1},0)$	$U(1)_Y$	0

Gauge coupling constants g_s (for quantum chromodynamics) and g (for weak isospin $SU(2)_L$) have been introduced following the procedure described in section 3.4. A third gauge coupling constant g', for weak isospin $U(1)_Y$ will appear later on. These three parameters (one for each simple or $U(1)$ factor in the gauge group) characterize all gauge boson interactions.

5.2 Quarks and leptons

Spinor fields of the Standard Model describe *quarks* and *leptons*. They differ by their transformations under the gauge group $SU(3) \times SU(2)_L \times U(1)_Y$ and by their interactions with the corresponding gauge bosons. Quarks have strong interactions, they couple to gluons, while leptons only feel weak and electromagnetic forces.

We first briefly discuss parity and its violation in fermion interactions.

[1] In the table, **8** is the adjoint representation of $SU(3)$, **3** is the adjoint representation of $SU(2)$, with $T_3 = +1, 0, -1$ and **1** is the singlet, invariant representation. For $U(1)_Y$, the $U(1)$ charge is indicated: gauge bosons never have $U(1)$ charges since $U(1)$ is abelian.

5.2.1 Parity

Parity transformation P is $(x^0 = t, \vec{x}) \longrightarrow (x^0, -\vec{x})$. In the Dirac operator,

$$\gamma^\mu \partial_\mu = \gamma^0 \partial_0 + \vec{\gamma} \cdot \vec{\nabla} \quad \longrightarrow \quad \gamma^0 \partial_0 - \vec{\gamma} \cdot \vec{\nabla} = \gamma^0 (\gamma^\mu \partial_\mu) \gamma^0.$$

Hence, if ψ is a solution of Dirac equation,

$$\psi_P = \gamma^0 \psi$$

is a solution of Dirac equation in parity-transformed coordinates. But is ψ is a left-handed (right-handed) Weyl spinor, then ψ_P is right-handed (left-handed). Hence, if parity is a symmetry, left- and right-handed spinors have identical interactions. For instance, consider the interaction of a gauge boson A_μ with a left-handed spinor. The Lagrangian term is proportional to

$$A_\mu (\overline{\psi}_L \gamma^\mu \psi_L) = A_\mu (\overline{\psi} \gamma^\mu P_L \psi), \qquad\qquad P_L = \frac{1}{2}(\mathbb{I} + \gamma_5)$$

Under parity, since gauge fields transform like coordinates, $A_\mu \to (A_0, -\vec{A})$, $\psi \to \psi_P = \gamma^0 \psi$ and then

$$A_\mu \, (\overline{\psi}_L \gamma^\mu \psi_L) \quad \longrightarrow \quad A_0 \, (\overline{\psi} \gamma^0 \gamma^0 P_L \gamma^0 \psi) - \vec{A} \cdot (\overline{\psi} \gamma^0 \vec{\gamma} P_L \gamma^0 \psi) = A_\mu \, (\overline{\psi}_R \gamma^\mu \psi_R).$$

Parity conservation implies then that spinors of both chiralities have identical gauge interactions. In other words, in a parity-conserving theory, Weyl spinors of both chiralities transform in the same representations.[2]

5.2.2 Quark and lepton representations

Since weak interactions violate parity, it is then natural to discuss the gauge transformations of Weyl spinors and it is simpler to use left-handed spinors only, turning right-handed spinors into their charge-conjugate left-handed spinors[3]:

$$\psi_R \qquad \longrightarrow \qquad (\psi_R)^c = C \gamma^0 (\psi_R^\dagger)^\tau = (\psi^c)_L.$$

Since charge conjugation involves a hermitian conjugation, it also conjugates gauge quantum numbers: it exchanges particles and antiparticles. We will then consider left-handed components of quarks, antiquarks, leptons and antileptons.

Quarks and leptons are classified according to their gauge transformations, according to their G_{SM} representation. Experiments and phenomenology suggest three basic rules:

[2] It is then simpler and sufficient to use Dirac spinors.

[3] Paragraph 1.4.1.

1. Quarks have strong interactions, they are colour triplets, *i.e.* they transform in representation $\mathbf{3}$ of $SU(3)$. Antiquarks transform in the conjugate representation $\overline{\mathbf{3}}$. Since leptons and antileptons do not feel strong interactions, they transform in the trivial representation $\mathbf{1}$ of $SU(3)$, in which all $SU(3)$ generators are the number zero.

2. Left-handed fermions (quarks and leptons) are $SU(2)_L$ doublets: they transfrom in representation $\mathbf{2}$ of $SU(2)_L$. Left-handed antifermions (antiquarks and antileptons), equivalent to right-handed fermions, are $SU(2)_L$ singlets.

3. The classification of fermions corresponds to a sequence of identical generations, considering their $SU(3) \times SU(2)_L \times U(1)_Y$ transformations. Generations differ by quark and lepton masses (and mixing angles). Observation has revealed three generations, and further generations (with light neutrinos as in the first three generations) are excluded by particle physics and cosmology.

These empirical rules lead to the following classification:

Particle	*Field*	*Representation*	*Electric charges*
Left-handed quarks	$\psi_Q^{(n)\,j\alpha}$	$(\mathbf{3}, \mathbf{2}, 1/6)$	2/3, −1/3
Left-handed antiquarks	$\psi_{U^c\,j}^{(n)}$	$(\overline{\mathbf{3}}, \mathbf{1}, -2/3)$	−2/3
	$\psi_{D^c\,j}^{(n)}$	$(\overline{\mathbf{3}}, \mathbf{1}, 1/3)$	1/3
Left-handed leptons	$\psi_L^{(n)\,\alpha}$	$(\mathbf{1}, \mathbf{2}, -1/2)$	0, −1
Left-handed antileptons	$\psi_{E^c}^{(n)}$	$(\mathbf{1}, \mathbf{1}, 1)$	1
	$\psi_{N^c}^{(n)}$	$(\mathbf{1}, \mathbf{1}, 0)$	0

$$\begin{aligned} n &= 1,2,3 \\ j &= 1,2,3 \\ \alpha &= 1,2 \end{aligned}$$

The notation is as follows: since all spinors are left-handed, the index L is systematically omitted. Since fermions are classified in three identical generations, a generation index $n = 1, 2, 3$ is introduced and in each generation fermions are characterized by indices Q, U^c, D^c, L, E^c, N^c related to the weak hypercharge quantum number. Then index j is an $SU(3)$ (colour) index. A superscript j indicates representation $\mathbf{3}$ (quarks), a subscript indicates the conjugate representation $\overline{\mathbf{3}}$ (antiquarks). Finally, index α indicates a $SU(2)$ doublet representation $\mathbf{2}$. The precise identification of these spinor fields with quarks u, d, s, c, b, t, and leptons e, μ, τ will be obtained below, after the full theory has been constructed.

The last column in the table includes quantum numbers (*i.e.* representations) under $SU(3)$, $SU(2)_L$ and $U(1)_Y$ of each fermion field. It indicates, for instance, that the left-handed quark doublet $\psi_Q^{(n)\,j\alpha}$ transforms as a colour $SU(3)$ triplet (representation

3), a $SU(2)_L$ doublet **2**, and that its weak hypercharge is $Y = 1/6$:

$$\delta\psi_Q^{(n)\,j\alpha} = i\,\omega^A (T_A^{\mathbf{3}})^j_k\, \psi_Q^{(n)\,k\alpha} + i\,\omega^a (T_a^{\mathbf{2}})^\alpha_\beta\, \psi_Q^{(n)\,j\beta} + \frac{1}{6} i\,\omega\, \psi_Q^{(n)\,j\alpha}, \tag{5.5}$$

for a transformation with infinitesimal parameters ω^A, ω^a and ω for respectively $SU(3)$, $SU(2)_L$ and $U(1)_Y$. Or, similarly, for the left-handed antiquark U^c in representation $(\overline{\mathbf{3}}, \mathbf{1}, -2/3)$,

$$\delta\psi^{(n)}_{U^c\,j} = -i\,\omega^A (T_A^{\mathbf{3}})^k_j\, \psi^{(n)}_{U^c\,k} - \frac{2}{3} i\,\omega\, \psi^{(n)}_{U^c\,j}. \tag{5.6}$$

With respect to transformation (5.5), the first negative sign is due to representation $\overline{\mathbf{3}}$, instead of **3**. By charge conjugation, the gauge variation of the right-handed quark U_R would then write

$$\delta\psi_{U,R}^{(n)\,j} = i\,\omega^A (T_A^{\mathbf{3}})^j_k\, \psi_{U,R}^{(n)\,k} + \frac{2}{3} i\,\omega\, \psi_{U,R}^{(n)\,j}. \tag{5.7}$$

The representation is now $(\mathbf{3}, \mathbf{1}, 2/3)$, which is the conjugate of $(\overline{\mathbf{3}}, \mathbf{1}, -2/3)$. Notice that the right-handed neutrino described by spinor field $\psi^{(n)}_{N^c}$ is gauge invariant, $\delta\psi^{(n)}_{N^c} = 0$: it does not feel strong or electroweak interactions and does not interact with gauge bosons. It does however appear in Yukawa interactions with scalar fields and then in neutrino masses.

Each generation of quarks and leptons in the Standard Model includes then sixteen left-handed Weyl spinors.

5.2.3 Absence of chiral anomalies

We have mentioned in section 3.9 that if left-handed fermions transform in the representation with generators T_A^ℓ of the gauge group, consistency of the quantum theory requires then that the numbers

$$d^\ell_{ABC} = T(\ell)^{-1}\,\mathrm{Tr}\Big(T_A^\ell \{T_B^\ell, T_C^\ell\}\Big)$$

vanish. To verify that this condition is indeed verified within each quark-lepton generation of the Standard Model, we need to study all possible choices of generators in the symmetric trace. We use the notation:

Indices A, B or C: generators of $SU(3)$;

Indices a, b or c: generators of $SU(2)_L$;

Indice Q: generator of the electric charge.

Since $Q = T^3 + Y$, using the weak hypercharge generator Y is not necessary. Some conditions are trivially verified:

1. Generators of $SU(2)_L$ and $SU(3)$ are traceless: $d^\ell_{aBC} = d^\ell_{aBQ} = d^\ell_{aQQ} = d^\ell_{Abc} = d^\ell_{AQQ} = 0$.
2. Strong and electromagnetic interactions conserve parity: $d^\ell_{ABC} = d^\ell_{ABQ} = d^\ell_{QQQ} = 0$.
3. Since $\mathrm{Tr}(\sigma_a\{\sigma_b,\sigma_c\}) = 2\delta_{bc}\,\mathrm{Tr}(\sigma_a) = 0$, $d^\ell_{abc} = 0$.

A single non-trivial condition remains then: $d^\ell_{abQ} = 0$. Under $SU(2)_L$, fermions are either doublets,

$$T^{\mathbf{2}}_a = \frac{1}{2}\sigma_a, \qquad \{T^{\mathbf{2}}_a, T^{\mathbf{2}}_b\} = \frac{1}{2}\delta_{ab}\,\mathbb{I},$$

or singlets $T^{\mathbf{1}}_a = 0$. Since $d^\ell_{abQ} \sim \mathrm{Tr}(Q\{T^\ell_a, T^\ell_b\})$, the condition $d^\ell_{abQ} = 0$ leads to

$$\sum_I Q_I = 0, \tag{5.8}$$

with index I running over left-handed fermions with weak isospin $T_3 = \pm\frac{1}{2}$ only. In a generation of quarks and leptons, the leptonic contribution to the trace is -1 (the charged lepton $\psi^{(n)}_{E,L}$). Quarks $\psi^{(n)\,j}_{U,L}$ and $\psi^{(n)\,j}_{D,L}$ contribute with

$$3\left(\frac{2}{3} - \frac{1}{3}\right) = 1,$$

with a factor 3 since each quark exists in three colours. Then, contributions of quarks and leptons cancel each other and each generation is anomaly-free.

The absence of chiral anomaly, imposed by consistency of the relativistic quantum field theory, is the only relation between quark and lepton properties present in the Standard Model. With the sequential structure in identical generations of quarks and leptons, it imposes the quantization of the weak hypercharge: if one normalizes $Y = -1/2$ on leptonic doublets, then the hypercharge of quarks must be $1/6$ to cancel the anomaly with three colours. Similarly, if charged leptons have electric charge -1 (choice of normalization), then quarks have electric charges $1/6 \pm 1/2 = 2/3$ or $-1/3$.

5.3 Scalar fields

Since weak interactions are mediated by massive gauge bosons $W^\pm$ and Z^0, scalar fields must be introduced to break $SU(2)_L \times U(1)_Y$ into the electromagnetic gauge symmetry

$U(1)_Q$, $Q = T_3 + Y$. With three broken symmetries and at least one physical Higgs boson, the minimal choice is a complex $SU(2)_L$ doublet including an electrically neutral field (which receives the vacuum expectation value breaking the symmetry). This last condition fixes the weak hypercharge of the scalar doublet to $Y = \pm 1/2$. Hence, the minimal choice used by the Standard Model has complex scalar fields H^α, $\alpha = 1, 2$, transforming under G_{SM} in representation

$$(\mathbf{1}, \mathbf{2}, -1/2).$$

Its infinitesinal gauge transformations are

$$\begin{aligned} SU(3): \quad & \delta H^\alpha &=& \ 0, \\ SU(2)_L: \quad & \delta H^\alpha &=& \ i\,\omega^a (T_a^{\mathbf{2}})^\alpha_\beta\, H^\beta, \\ U(1)_Y: \quad & \delta H^\alpha &=& \ -\tfrac{1}{2} i\omega H^\alpha. \end{aligned} \tag{5.9}$$

It is useful to define

$$H = \begin{pmatrix} H^1 \\ H^2 \end{pmatrix}, \qquad \overline{H} = i\sigma^2\, H^* = \begin{pmatrix} H^{2*} \\ -H^{1*} \end{pmatrix}. \tag{5.10}$$

In fact, since $T_a^{\mathbf{2}} = \frac{1}{2}\sigma^a$ and $\sigma^{a*} = i\sigma^2\,\sigma^a\, i\sigma^2$, we obtain

$$\begin{aligned} SU(3): \quad & \delta \overline{H}^\alpha &=& \ 0, \\ SU(2)_L: \quad & \delta \overline{H}^\alpha &=& \ i\,\omega^a (T_a^{\mathbf{2}})^\alpha_\beta\, \overline{H}^\beta, \\ U(1)_Y: \quad & \delta \overline{H}^\alpha &=& \ +\tfrac{1}{2} i\omega \overline{H}^\alpha. \end{aligned} \tag{5.11}$$

Quantum numbers of the conjugate $\overline{H}^\alpha$ are then

$$(\mathbf{1}, \mathbf{2}, +1/2).$$

The quantity

$$\epsilon_{\alpha\beta}\overline{H}^\alpha H^\beta = H^{1*}H^1 + H^{2*}H^2 = H^\dagger H, \qquad \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \quad \epsilon_{12} = 1,$$

is an invariant of G_{SM} which will be useful in the construction of the Lagrangian density. In fact, any invariant function of H and $\overline{H}$ is actually a function of this unique invariant.

5.4 Covariant derivatives and Lagrangian density

According to the procedure described in sections 3.2 and 3.5, quantum numbers of fermion and scalar fields allow to write their covariant derivatives[4]. For a multiplet

[4] See equation (3.22).

of fields ϕ (spinors or scalars) in the representation $(\mathbf{n_3}, \mathbf{n_2}, Y)$ of G_{SM}, the general expression is:

$$D_\mu \phi = \partial_\mu \phi - ig_s\, A_\mu^A (T_A^{\mathbf{n_3}} \phi) - ig\, W_\mu^a (T_a^{\mathbf{n_2}} \phi) - ig'\, Y\, B_\mu \phi, \tag{5.12}$$

with $T_A^{\mathbf{n_3}} = 0$ if the $SU(3)$ representation is $\mathbf{n_3} = \mathbf{1}$, $T_A^{\mathbf{n_3}} = -T_A^{\mathbf{3}}$ if $\mathbf{n_3} = \overline{\mathbf{3}}$ and $T_a^{\mathbf{n_2}} = 0$ if the $SU(2)_L$ representation is $\mathbf{n_2} = \mathbf{1}$. It is admitted that $(T_A^{\mathbf{3}}\phi)$ and $(T_a^{\mathbf{2}}\phi)$ include the appropriate sum on $SU(3)$ and $SU(2)_L$ indices. Explicitly, we obtain

$$\begin{array}{rcl}
D_\mu \psi_Q^{(n)\,j\alpha} &=& \partial_\mu \psi_Q^{(n)\,j\alpha} - ig_s A_\mu^A (T_A^{\mathbf{3}})^j_k \psi_Q^{(n)\,k\alpha} - igW_\mu^a (T_a^{\mathbf{2}})^\alpha_\beta \psi_Q^{(n)\,j\beta} - \frac{1}{6} ig' B_\mu \psi_Q^{(n)\,j\alpha}, \\
D_\mu \psi^{(n)}_{U^c\,j} &=& \partial_\mu \psi^{(n)}_{U^c\,j} + ig_s A_\mu^A (T_A^{\mathbf{3}})^k_j \psi^{(n)}_{U^c\,k} + \frac{2}{3} ig' B_\mu \psi^{(n)}_{U^c\,j}, \\
D_\mu \psi^{(n)}_{D^c\,j} &=& \partial_\mu \psi^{(n)}_{D^c\,j} + ig_s A_\mu^A (T_A^{\mathbf{3}})^k_j \psi^{(n)}_{D^c\,k} - \frac{1}{3} ig' B_\mu \psi^{(n)}_{D^c\,j}, \\
D_\mu \psi_L^{(n)\,\alpha} &=& \partial_\mu \psi_L^{(n)\,\alpha} - igW_\mu^a (T_a^{\mathbf{2}})^\alpha_\beta \psi_L^{(n)\,\beta} + \frac{1}{2} ig' B_\mu \psi_L^{(n)\,\alpha}, \\
D_\mu \psi^{(n)}_{E^c} &=& \partial_\mu \psi^{(n)}_{E^c} - ig' B_\mu \psi^{(n)}_{E^c}, \\
D_\mu \psi^{(n)}_{N^c} &=& \partial_\mu \psi^{(n)}_{N^c}, \\
D_\mu H^\alpha &=& \partial_\mu H^\alpha - igW_\mu^a (T_a^{\mathbf{2}})^\alpha_\beta H^\beta + \frac{1}{2} ig' B_\mu H^\alpha, \\
D_\mu \overline{H}^\alpha &=& \partial_\mu \overline{H}^\alpha - igW_\mu^a (T_a^{\mathbf{2}})^\alpha_\beta \overline{H}^\beta - \frac{1}{2} ig' B_\mu \overline{H}^\alpha.
\end{array} \tag{5.13}$$

With these expressions, the Lagrangian density of the Standard Model is of the form

$$\mathcal{L} = \mathcal{L}_{cin.} + \mathcal{L}_{Yuk.} - V(H, H^\dagger) + \mathcal{L}_{N^c}. \tag{5.14}$$

The first three terms respectively include kinetic and gauge interactions of all fields, fermion–scalar Yukawa interactions and the potential for scalar fields. The last term is a contribution specific to the right-handed neutrinos, which are gauge-invariant. All gauge boson interactions follow from covariantization of derivatives [eqs. (5.13)] of from the gauge curvatures (5.4):

$$\begin{array}{rcl}
\mathcal{L}_{cin.} &=& -\frac{1}{4} F_{\mu\nu}^A F^{\mu\nu\,A} - \frac{1}{4} W_{\mu\nu}^a W^{\mu\nu\,a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\
&& + i \displaystyle\sum_{n=1}^{3} \Big[\overline{\psi}^{(n)}_{Q\,j\alpha} \gamma^\mu D_\mu \psi_Q^{(n)\,j\alpha} + \overline{\psi}^{(n)\,j}_{U^c} \gamma^\mu D_\mu \psi^{(n)}_{U^c\,j} + \overline{\psi}^{(n)\,j}_{D^c} \gamma^\mu D_\mu \psi^{(n)}_{D^c\,j} \\
&& \qquad + \overline{\psi}^{(n)}_{L\,\alpha} \gamma^\mu D_\mu \psi_L^{(n)\,\alpha} + \overline{\psi}^{(n)}_{E^c} \gamma^\mu D_\mu \psi^{(n)}_{E^c} + \overline{\psi}^{(n)}_{N^c} \gamma^\mu \partial_\mu \psi^{(n)}_{N^c} \Big] \\
&& + (D_\mu H)^\dagger (D^\mu H).
\end{array} \tag{5.15}$$

Yukawa interactions are obtained by writing all gauge-invariant fermion–fermion–scalar terms. The contributions involving electrically charged particles are

$$\begin{aligned}\mathcal{L}_{Yuk.} &= -\sum_{n,m=1}^{3}\Big[\lambda_D^{nm}\epsilon_{\alpha\beta}H^\alpha(\psi^{(n)}_{D^c\,j})^\tau C\psi_Q^{(m)\,j\beta} + \lambda_U^{nm}\epsilon_{\alpha\beta}\overline{H}^\alpha(\psi^{(n)}_{U^c\,j})^\tau C\psi_Q^{(m)\,j\beta} \\ &\quad +\lambda_E^{nm}\epsilon_{\alpha\beta}H^\alpha(\psi^{(n)}_{E^c})^\tau C\psi_L^{(m)\,\beta}\Big] \;+\; \text{hermitian conjugate.}\end{aligned} \tag{5.16}$$

Since for any spinor[5] $\psi_L{}^\tau C = \overline{\psi}^c_R$, these interactions can be rewritten as

$$\begin{aligned}\mathcal{L}_{Yuk.} &= -\sum_{n,m=1}^{3}\Big[\lambda_D^{nm}\epsilon_{\alpha\beta}H^\alpha\,(\overline{\psi}^{(n)}_{D,R\,j}\psi^{(m)\,j\beta}_{Q,L}) + \lambda_U^{nm}\epsilon_{\alpha\beta}\overline{H}^\alpha\,(\overline{\psi}^{(n)}_{U,R\,j}\psi^{(m)\,j\beta}_{Q,L}) \\ &\quad +\lambda_E^{nm}\epsilon_{\alpha\beta}H^\alpha\,(\overline{\psi}^{(n)}_{E,R}\psi^{(m)\,\beta}_{L,L})\Big] \;+\; \text{hermitian conjugate,}\end{aligned} \tag{5.17}$$

reintroducing chirality indices L and R. Since an invariant of G_{SM} of the form fermion–fermion does not exist for charged particles, the Lagrangian does not include any mass term for these fields. Such terms are forbidden by the fact that $SU(2)_L$ only transforms left-handed fields. This is however only true as long as the symmetry is not spontaneously broken. The Higgs mechanism which generates $W^\pm$ and Z^0 masses also produces quark and lepton masses.

The scalar potential is the most general fourth-order polynomial in H and $H^\dagger$ (or $\overline{H}$) left invariant by gauge transformations of G_{SM}:

$$V(H,H^\dagger) = -\mu^2 H^\dagger H + \frac{1}{2}\lambda(H^\dagger H)^2. \tag{5.18}$$

At this stage, the theory includes the (dimensionless) coupling constants

$$g_s, g, g', \lambda_U^{nm}, \lambda_D^{nm}, \lambda_E^{nm}, \lambda,$$

which are arbitrary parameters of the theory. It also includes an arbitrary mass scale μ. Notice that Yukawa couplings are complex numbers: some of the phases can however be eliminated by redefining the phases of scalar and fermion fields.

The missing contributions of the right-handed neutrino spinors, $\psi^{(n)}_{N,R}$ (or the left-handed antineutrinos $\psi^{(n)}_{N^c,L}$) are:

$$\begin{aligned}\mathcal{L}_{N^c} &= -\frac{1}{2}\sum_{n,m=1}^{3} M_{nm}\,(\psi^{(n)}_{N^c,L})^\tau C\psi^{(m)}_{N^c,L} \;+\; \text{hermitian conjugate} \\ &\quad -\sum_{n,m=1}^{3}\lambda_N^{nm}\epsilon_{\alpha\beta}\overline{H}^\alpha\overline{\psi}^{(n)}_{N,R}\psi^{(m)\,\beta}_{L,L} \;+\; \text{hermitian conjugate.}\end{aligned} \tag{5.19}$$

[5] Paragraph 1.4.1.

Since the right-handed neutrino is gauge invariant [representation $(\mathbf{1}, \mathbf{1}, 0)$], it does not have gauge interactions and its covariant derivative is an ordinary derivative ∂_μ: it then cannot be directly detected in a strong or electroweak interaction process. However, the gauge invariance of these fields allows the first term in contribution (5.19), the *Majorana mass terms* of right-handed neutrinos. The second line in expression (5.19) describes Yukawa interactions of right-handed neutrinos with leptons and scalars. Via the Higgs mechanism, $\mathcal{L}_{N^c}$ generates the mass matrix of neutrinos which, because of the presence of Majorana masses, has a more complicated structure with more parameters than charged lepton or quark mass matrices.

Chapter 6

Symmetry breaking and gauge boson masses

6.1 Higgs mechanism and unitary gauge

The spontaneous symmetry breaking of $SU(2)_L \times U(1)_Y$ into $U(1)_Q$ with a complex scalar doublet H has been discussed in section 4.3. We use here the results. In the unitary gauge where the three Goldstone bosons have been "gauged away", one can choose

$$H_{unit.} = \frac{1}{\sqrt{2}} \begin{pmatrix} h(x) + v \\ 0 \end{pmatrix}, \qquad v^2 = \frac{2\mu^2}{\lambda}, \tag{6.1}$$

assuming that the parameters of the potential (4.39) verify $\mu^2 > 0$, $\lambda > 0$. The real scalar field $h(x)$ describes the Higgs boson. The vacuum expectation value $\langle H \rangle$ breaks $SU(2)_L \times U(1)_Y$, leaving invariant the subgroup $U(1)_Q$ of the electric charge, generated by

$$Q = T^3 + Y. \tag{6.2}$$

Since the scalar doublet H is invariant under transformations of the gauge group of quantum chromodynamics $SU(3)$, the subgroup of G_{SM} left invariant by the expectation value $\langle H \rangle$ is $SU(3) \times U(1)_Q$. This group is the exact gauge symmetry of the Standard Model.

The Lagrangian density in the unitary gauge is obtained by substituting expression (6.1) of the scalar doublet in contributions (5.14), (5.15), (5.17) and (5.18). The correct, physically-relevant definition of the fields is then found by performing redefinitions which diagonalize the mass matrices of spinors and gauge fields. The unitary gauge Lagrangian will be split in four parts,

$$\mathcal{L}_{unit.} = \mathcal{L}_{bos.} + \mathcal{L}_{ferm.,1} + \mathcal{L}_{ferm.,2} - V(h), \tag{6.3}$$

Field	G_{SM}	T_3	Y	$Q = T_3 + Y$
A^A_μ	$(\mathbf{8},\, \mathbf{1},\, 0)$	0	0	0
W^a_μ	$(\mathbf{1},\, \mathbf{3},\, 0)$	$1, -1, 0$	0	$1,\, -1,\, 0$
B_μ	$(\mathbf{1},\, \mathbf{1},\, 0)$	0	0	0
h		$1/2$	$-1/2$	0
$\psi^{(n)\,j\alpha}_Q = \begin{pmatrix} \psi^{(n)\,j}_U \\ \psi^{(n)\,j}_D \end{pmatrix}$	$(\mathbf{3}, \mathbf{2}, 1/6)$	$1/2$ $-1/2$	$1/6$	$2/3$ $-1/3$
$\psi^{(n)}_{U^c\,j}$	$(\overline{\mathbf{3}}, \mathbf{1}, -2/3)$	0	$-2/3$	$-2/3$
$\psi^{(n)}_{D^c\,j}$	$(\overline{\mathbf{3}}, \mathbf{1}, 1/3)$	0	$1/3$	$1/3$
$\psi^{(n)\,\alpha}_L = \begin{pmatrix} \psi^{(n)}_N \\ \psi^{(n)}_E \end{pmatrix}$	$(\mathbf{1}, \mathbf{2}, -1/2)$	$1/2$ $-1/2$	$-1/2$	0 -1
$\psi^{(n)}_{E^c}$	$(\mathbf{1}, \mathbf{1}, 1)$	0	1	1
$\psi^{(n)}_{N^c}$	$(\mathbf{1}, \mathbf{1}, 0)$	0	0	0

Table 6.1: Fields and quantum numbers

describing respectively the kinetic terms and gauge interactions of bosons, the fermion mass terms, fermion interactions and kinetic terms and finally the scalar potential of the Higgs field. In the unitary gauge, the scalar potential (5.18) becomes[1]

$$V(h) = -\frac{v^2}{4} + \mu^2 h^2 + \frac{\lambda}{2} v h^3 + \frac{\lambda}{8} h^4. \tag{6.4}$$

Hence, the mass of the Higgs boson is given by the (arbitrary) parameter μ,

$$m_h = \sqrt{2}\,\mu = \sqrt{\lambda}\,v, \tag{6.5}$$

and its self-interactions are controlled by coupling constant λ.

The quantum numbers under G_{SM} and the electric charges of the fields in the unitary gauge (*i.e.* using expression (6.1) for the scalar doublet H) are listed in table 6.1. Fermions are left-handed Weyl spinors. Each quark-lepton generation includes the three colours and the four components (Dirac spinor) of a quark $U^{(n)}$ with charge $2/3$ and of a quark $D^{(n)}$ with charge $-1/3$, the four components of a lepton $E^{(n)}$ with charge -1, as well as the four components of a neutrino $N^{(n)}$. The right-handed chirality of the neutrino is a singlet under G_{SM}: it does not have any strong or electroweak interaction.

[1] The first constant contribution is irrelevant and can be discarded.

6.1.1 Gauge bosons

The bosonic contribution $\mathcal{L}_{bos.}$ has already been derived in section 4.3. It is obtained from

$$\mathcal{L}_{bos.} = -\frac{1}{4}F^A_{\mu\nu}F^{\mu\nu\,A} - \frac{1}{4}W^a_{\mu\nu}W^{\mu\nu\,a} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + (D_\mu H)^\dagger(D^\mu H) \tag{6.6}$$

by replacing $D_\mu H$ by its expression in the unitary gauge,

$$D_\mu H = \frac{1}{\sqrt{2}}\begin{pmatrix} \partial_\mu h - \frac{i}{2}\left(gW^3_\mu - g'B_\mu\right)(h+v) \\ -\frac{ig}{\sqrt{2}}\left(W^1_\mu + iW^2_\mu\right)(h+v) \end{pmatrix}.$$

The result is given in eqs. (4.54) and (4.55), in terms of the new vector fields

$$\begin{array}{rclrcl} W^+_\mu &=& \frac{1}{\sqrt{2}}(W^1_\mu - iW^2_\mu), & W^-_\mu &=& \frac{1}{\sqrt{2}}(W^1_\mu + iW^2_\mu) \;=\; (W^+_\mu)^\dagger, \\ Z_\mu &=& \cos\theta_W W^3_\mu - \sin\theta_W B_\mu, & A_\mu &=& \sin\theta_W W^3_\mu + \cos\theta_W B_\mu, \end{array} \tag{6.7}$$

and of the *weak mixing angle*, or *Weinberg angle*, θ_W, defined by

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}. \tag{6.8}$$

The massless field A_μ is the gauge field of the exact symmetry $U(1)_Q$, the photon field. This can be seen, besides the simple fact that it is the the only $SU(2)_L \times U(1)_Y$ gauge field left massless, by observing that covariant derivatives (5.12) include the combination

$$gW^3_\mu T^3 + g'B_\mu Y = g\sin\theta_W\left[A_\mu Q + Z_\mu(\operatorname{cotg}\theta_W T^3 - \operatorname{tg}\theta_W Y)\right].$$

The field A_μ couples then to the electric charge Q and the coupling constant of the electromagnetic interaction is

$$e = g\sin\theta_W = \frac{gg'}{\sqrt{g^2+g'^2}}. \tag{6.9}$$

Gauge interactions in eq. (4.55) correspond to the particular case $Y = 0$ since these fields do not have weak hypercharge. Each field A_μ is associated with coupling constant $e = g\sin\theta_W$, while Z_μ is associated with $g\sin\theta_W\operatorname{cotg}\theta_W = g\cos\theta_W$. In contrast, for the Higgs boson, $T^3 = 1/2 = -Y$; the coupling constant associated with each Z_μ is then

$$\frac{1}{2}g\sin\theta_W\left[\operatorname{cotg}\theta_W + \operatorname{tg}\theta_W\right] = \frac{g}{2\cos\theta_W},$$

which appears in the last interaction in expression (4.55). Cubic and quartic gluon interactions are controlled by the strong coupling constant g_s while the interaction with four $W^{\pm}$ is of order g^2. The Higgs boson has interactions with W^+W^- and Z^0Z^0. Knowledge of the $W^{\pm}$ mass determines the combination

$$M_W = \frac{1}{2} g v \,, \tag{6.10}$$

while relation[2]

$$M_W = \cos\theta_W \, M_Z \tag{6.11}$$

follows from the use of a scalar doublet to break symmetry $SU(2)_L \times U(1)_Y$. Measuring M_W, M_W/M_Z and e allows then to determine g, g' and v. If $m_h > 2M_W = gv$, the Higgs boson decays into W^+W_- pairs, if $m_h > 2M_Z$, is also decays into Z^0Z^0 pairs.

[2] This relation receives perturbative quantum corrections.

Chapter 7

Quarks and leptons: masses and interactions

7.1 Fermion masses

Since $SU(2)_L$ only transforms left-handed quark and lepton fields, fermion mass terms which couple left- and right-handed spinors are forbidden by gauge invariance. However, Yukawa interactions with the Higgs doublet H generate fermion masses as soon as $SU(2)_L$ is spontaneously broken by the vacuum expectation value $\langle H \rangle$.

We will discuss neutrino masses, which are complicated by the presence of Majorana terms in (5.19), in a separate section[1]. Hence, in this section, we keep neutrinos massless: this was the case in the original formulation of the Standard Model and, until recently (1998), massless neutrinos were compatible with observations.

The fermion mass terms for quarks and charged leptons induced by $\langle H \rangle$ are then

$$\begin{aligned} \mathcal{L}_{ferm.,1} &= -\frac{v}{\sqrt{2}} \sum_{n,m=1}^{3} \Big[\lambda_D^{nm} (\overline{\psi}^{(n)}_{D,Rj} \psi^{(m)j}_{D,L}) + \lambda_U^{nm} (\overline{\psi}^{(n)}_{U,Rj} \psi^{(m)j}_{U,L}) \\ &\qquad + \lambda_E^{nm} (\overline{\psi}^{(n)}_{E,R} \psi^{(m)}_{E,L}) \Big] + \text{hermitian conjugate.} \end{aligned} \tag{7.1}$$

In this basis of the spinor fields, mass matrices are proportional to the Yuwawa coupling matrices $\lambda_{D,U,E}$ and are not in general diagonal in generation indices n, m. Spinor kinetic terms are actually invariant under independent unitary rotations of the left-handed spinors of a given electric charge[2] and this freedom can be used to diagonalize

[1] Section 7.3.

[2] See eq. (5.15).

the mass matrix:

$$\begin{aligned}
\psi^{(n)j}_{D,L} &= (\mathcal{U}_D)^n_m \Psi^{(m)j}_{D,L}, & \psi^{(n)j}_{D,R} &= (\mathcal{O}_D)^n_m \Psi^{(m)j}_{D,R}, \\
\psi^{(n)j}_{U,L} &= (\mathcal{U}_U)^n_m \Psi^{(m)j}_{U,L}, & \psi^{(n)j}_{U,R} &= (\mathcal{O}_U)^n_m \Psi^{(m)j}_{U,R}, \\
\psi^{(n)}_{E,L} &= (\mathcal{U}_E)^n_m \Psi^{(m)}_{E,L}, & \psi^{(n)}_{E,R} &= (\mathcal{O}_E)^n_m \Psi^{(m)}_{E,R}.
\end{aligned} \tag{7.2}$$

The unitary matrices $\mathcal{U}_{D,U,E}$ and $\mathcal{O}_{D,U,E}$ are chosen[3] such that matrices $\mathcal{O}^\dagger_D \lambda_D \mathcal{U}_D$, $\mathcal{O}^\dagger_U \lambda_U \mathcal{U}_U$ and $\mathcal{O}^\dagger_E \lambda_E \mathcal{U}_E$ are diagonal with real eigenvalues:

$$\begin{aligned}
(\mathcal{O}^\dagger_D \lambda_D \mathcal{U}_D)^{nm} &= \lambda_{D,n}\, \delta^{nm}, \\
(\mathcal{O}^\dagger_U \lambda_U \mathcal{U}_U)^{nm} &= \lambda_{U,n}\, \delta^{nm}, \qquad\qquad (n, m = 1, 2, 3). \\
(\mathcal{O}^\dagger_E \lambda_E \mathcal{U}_E)^{nm} &= \lambda_{E,n}\, \delta^{nm}.
\end{aligned} \tag{7.3}$$

Since neutrinos are massless (in this section), a redefinition of spinors $\psi^{(n)}_{N,L}$ or $\psi^{(n)}_{N,R}$ is not needed. Conventionally, eigenvalues are ordered according to

$$\lambda_{\sharp,1} < \lambda_{\sharp,2} < \lambda_{\sharp,3}, \qquad\qquad \sharp = D, U, E.$$

With these manipulations, fermion mass terms (7.1) become simply

$$\begin{aligned}
\mathcal{L}_{ferm.,1} = -\sum_{n=1}^{3} \Big[& m_{D,n}(\overline{\Psi}^{(n)}_{D,Rj} \Psi^{(n)j}_{D,L}) + m_{U,n}(\overline{\Psi}^{(n)}_{U,Rj} \Psi^{(n)j}_{U,L}) + m_{E,n}(\overline{\Psi}^{(n)}_{E,R} \Psi^{(n)}_{E,L}) \Big] \\
& + \text{hermitian conjugate.}
\end{aligned} \tag{7.4}$$

It is in this new basis of eigenstates of fermion mass matrices that spinorial fields are identified with quarks u, d, s, c, b, t and charged leptons e, μ, τ. The first generation $n = 1$ includes the lightest charged fermions e, d and u, and a first neutrino ν_e:

$$m_e = m_{E,1} = \lambda_{E,1} \frac{v}{\sqrt{2}}, \qquad m_d = m_{D,1} = \lambda_{D,1} \frac{v}{\sqrt{2}}, \qquad m_u = m_{U,1} = \lambda_{U,1} \frac{v}{\sqrt{2}},$$

The second generation $n = 2$ includes μ, s, c and ν_μ, the third and last ($n = 3$) τ, b, t and ν_τ.

Quark and charged lepton masses are given by the eigenvalues of the Yukawa coupling matrices multiplied by $v/\sqrt{2}$: they are then arbitrary parameters: the Standard Model does not predict any relation between fermions masses.

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[3] It is always possible.

7.2 Fermion interactions and kinetic terms

Diagonalization of their mass matrices, as done in previous section, defines the physically-relevant basis of spinor fields. Their interactions are then of two types: firstly gauge interactions contained in covariant derivatives in expression (5.15) and naturally associated with propagation terms of spinor fields, secondly Yukawa interactions with the Higgs boson derived from eq. (5.17). We then write

$$\mathcal{L}_{ferm.,2} = \mathcal{L}_{gauge} + \mathcal{L}_{Yuk.} \tag{7.5}$$

to distinguish both types of interactions.

Since the unbroken gauge symmetry group is $SU(3)\times U(1)_Q$, one can certainly write fermion kinetic terms using covariant derivatives of this group. And since interactions dictated by this gauge symmetry (strong and electromagnetic interactions) respect parity, they can be formulated in terms of four-component Dirac spinors:

$$\begin{aligned}\mathcal{L}_{gauge} &= \sum_{n=1}^{3}\Big[i\overline{\Psi}^{(n)}_{Dj}\gamma^\mu \widetilde{D}_\mu \Psi^{(n)j}_D + i\overline{\Psi}^{(n)}_{Uj}\gamma^\mu \widetilde{D}_\mu \Psi^{(n)j}_U + i\overline{\Psi}^{(n)}_{E}\gamma^\mu \widetilde{D}_\mu \Psi^{(n)}_E \\ &\qquad + i\overline{\Psi}^{(n)}_{N,L}\gamma^\mu \partial_\mu \Psi^{(n)}_{N,L}\Big] + \mathcal{L}_{WZ}.\end{aligned}$$

$\mathcal{L}_{WZ}$ contains all fermion weak interactions with massive gauge bosons $W^\pm_\mu$ and Z_μ and covariant derivatives of $SU(3)\times U(1)_Q$ are

$$\begin{aligned}\widetilde{D}_\mu \Psi^{(n)j}_D &= \partial_\mu \Psi^{(n)j}_D - ig_s A^A_\mu (T^A_{\mathbf{3}})^j_k \Psi^{(n)k}_D + \tfrac{1}{3} ieA_\mu \Psi^{(n)j}_D, \\ \widetilde{D}_\mu \Psi^{(n)j}_U &= \partial_\mu \Psi^{(n)j}_U - ig_s A^A_\mu (T^A_{\mathbf{3}})^j_k \Psi^{(n)k}_U - \tfrac{2}{3} ieA_\mu \Psi^{(n)j}_U, \\ \widetilde{D}_\mu \Psi^{(n)}_E &= \partial_\mu \Psi^{(n)}_E + ieA_\mu \Psi^{(n)}_E,\end{aligned}$$

with again $e = g\sin\theta_W$. The electromagnetic interaction is of the form

$$eA_\mu\, j^\mu_{e.m.}, \qquad j^\mu_{e.m.} = \sum_{n=1}^{3}\Big[-\overline{\Psi}^{(n)}_E\gamma^\mu\Psi^{(n)}_E + \frac{2}{3}\overline{\Psi}^{(n)}_{Uj}\gamma^\mu\Psi^{(n)j}_U - \frac{1}{3}\overline{\Psi}^{(n)}_{Dj}\gamma^\mu\Psi^{(n)j}_D\Big],$$

as already obtained in section 1.4.

Similarly, the weak interaction of fermions has the following form:

$$\mathcal{L}_{WZ} = \frac{g}{\sqrt{2}}W^+_\mu j^{-\mu} + \frac{g}{\sqrt{2}}W^-_\mu j^{+\mu} + gZ_\mu j^{0\mu}. \tag{7.6}$$

The charged fermionic currents are

$$\begin{aligned}j^{-\mu} &= \sum_{n=1}^{3}\overline{\Psi}^{(n)}_{N,L}\gamma^\mu\Psi^{(n)}_{E,L} + \sum_{m,n=1}^{3}(\mathcal{U})^{mn}\overline{\Psi}^{(m)}_{U,Lj}\gamma^\mu\Psi^{(n)j}_{D,L}, \\ j^{+\mu} &= \sum_{n=1}^{3}\overline{\Psi}^{(n)}_{E,L}\gamma^\mu\Psi^{(n)}_{N,L} + \sum_{m,n=1}^{3}(\mathcal{U}^\dagger)^{mn}\overline{\Psi}^{(m)}_{D,Lj}\gamma^\mu\Psi^{(n)j}_{U,L} = [j^{-\mu}]^\dagger.\end{aligned} \tag{7.7}$$

The charged leptonic current defines the natural basis of the neutrino spinor fields $\Psi^{(n)}_{N,L}$. We have defined

$$\psi^{(n)}_{N,L} = (\mathcal{U}_E)^n_m \Psi^{(m)}_{N,L} \tag{7.8}$$

and $\Psi^{(n)}_{N,L}$ is then the left-handed neutrino associated with charged lepton $\Psi^{(n)}_E$: it is coupled to the charged lepton $\Psi^{(n)}_E$ by the weak interaction of the charged boson $W^\pm$. This is a legitimate definition as long as neutrinos are massless. With massless neutrinos, there is a conserved *leptonic number* for each generation.

In contrast, there exists a mixing between generations in charged current weak interactions of quarks, through the matrix

$$\mathcal{U}^{mn} = (\mathcal{U}_U^\dagger \mathcal{U}_D)^{mn}, \tag{7.9}$$

which is the *Cabibbo-Kobayashi-Maskawa matrix*. Since $\mathcal{U}$ is unitary, $\mathcal{U}^\dagger\mathcal{U} = I$, it contains in principle nine real parameters, including three rotation angles in the space of the three generations. It is however possible to redefine the six phases of spinors $\Psi^{(n)j}_{U,L}$ and $\Psi^{(n)j}_{D,L}$ without affecting the diagonal structure of quark mass matrices or the matrices governing other interactions[4]. Since a global phase does not act on $\mathcal{U}$, five of the nine parameters of $\mathcal{U}$ are unobservable. As a result, the Cabibbo-Kobayashi-Maskawa matrix includes three angles describing the weak coupling of quarks from different generations and one (observable) complex phase. This phase is a source for CP violation in charged current weak interactions.[5] Indeed, the transformation under CP of the hadronic term contained in $W^+_\mu j^{-\mu}$ is

$$\sum_{m,n=1}^{3} W^+_\mu (\mathcal{U})^{mn} \overline{\Psi}^{(m)}_{U,Lj} \gamma^\mu \Psi^{(n)j}_{D,L} \qquad \xrightarrow{CP} \qquad \sum_{m,n=1}^{3} W^-_\mu (\mathcal{U}^\tau)^{mn} \overline{\Psi}^{(m)}_{D,Lj} \gamma^\mu \Psi^{(n)j}_{U,L}.$$

The transformed term differs from $W^-_\mu j^{+\mu}$ if the Cabibbo-Kobayashi-Maskawa matrix is complex: $\mathcal{U}^\tau \neq \mathcal{U}^\dagger$. Even if it manifests itself in the quarks–$W^\pm$ interaction, the origin of the CP violation is to be found in the form of Yukawa couplings λ^{mn}_D and λ^{mn}_U. From their diagonalization (7.3) follows actually the form of the matrix $\mathcal{U}$ and then the value of a complex phase violating CP.

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[4]It is sufficient to apply the opposite redefinition to spinors $\Psi^{(n)j}_{U,R}$ and $\Psi^{(n)j}_{D,R}$ which do not appear in charged curents.

[5]With two generations only, $\mathcal{U}$ would have four parameters, an angle and three phases. Since one can eliminate $4-1$ phases in this case, the only observable parameter is the *Cabibbo angle* measuring the mixing of the two generations and CP symmetry cannot be violated with two generations only.

The neutral current appearing in the interaction fermions–Z_μ is

$$\begin{aligned} j^0_\mu \;=\; & \frac{1}{\cos\theta_W}\sum_{n=1}^{3}\Big[\frac{1}{2}\overline{\Psi}^{(n)}_{N,L}\gamma^\mu\Psi^{(n)}_{N,L} \\ & +\Big(-\frac{1}{2}+\sin^2\theta_W\Big)\,\overline{\Psi}^{(n)}_{E,L}\gamma^\mu\Psi^{(n)}_{E,L}+\sin^2\theta_W\,\overline{\Psi}^{(n)}_{E,R}\gamma^\mu\Psi^{(n)}_{E,R} \\ & +\Big(\frac{1}{2}-\frac{2}{3}\sin^2\theta_W\Big)\,\overline{\Psi}^{(n)}_{U,Lj}\gamma^\mu\Psi^{(n)j}_{U,L}-\frac{2}{3}\sin^2\theta_W\,\overline{\Psi}^{(n)}_{U,Rj}\gamma^\mu\Psi^{(n)j}_{U,R} \\ & +\Big(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_W\Big)\,\overline{\Psi}^{(n)}_{D,Lj}\gamma^\mu\Psi^{(n)j}_{D,L}+\frac{1}{3}\sin^2\theta_W\,\overline{\Psi}^{(n)}_{D,Rj}\gamma^\mu\Psi^{(n)j}_{D,R}\Big]. \end{aligned} \tag{7.10}$$

The neutral current weak interation does not allow generation mixing: there are no *flavour-changing neutral currents* (FCNC).

In the unitary gauge and in the basis where fermion mass matrices are diagonal, Yukawa interactions are obtained according to (6.1) by replacing v by h in $\mathcal{L}_{ferm.,1}$. We then obtain:

$$\begin{aligned} \mathcal{L}_{Yuk.} = -\frac{1}{\sqrt{2}}h\sum_{n=1}^{3}\Big[\lambda_{D,n}(\overline{\Psi}^{(n)}_{D,Rj}\Psi^{(n)j}_{D,L})+\lambda_{U,n}(\overline{\Psi}^{(n)}_{U,Rj}\Psi^{(n)j}_{U,L})+\lambda_{E,n}(\overline{\Psi}^{(n)}_{E,R}\Psi^{(n)}_{E,L})\Big] \\ + \text{hermitian conjugate,} \end{aligned} \tag{7.11}$$

as a function of eigenvalues (7.3). The coupling constant of a fermion with mass m_f to the Higgs boson is then

$$\frac{m_f}{v} = \frac{g}{2}\frac{m_f}{M_W}.$$

It is proportional to the fermion mass. It is a weak interaction (constant g) with a suppression factor $m_f/M_W \ll 1$ for all fermions, except for the top quark.

7.3 Massive neutrinos

Neutrinos are not massless, but their masses are nevertheless much smaller than quarks and charged lepton masses. There is a crucial difference: right-handed neutrinos are G_{SM} singlets $(\mathbf{1},\mathbf{1},0)$ and a Majorana mass term is then allowed.

We first consider the simple model of a single generation. We then have two neutrino Weyl spinors $\psi_{N,L}$ and $\psi_{N,R}$ and two mass terms are allowed:

$$\begin{aligned} & -m(\overline{\psi}_{N,L}\psi_{N,R}+\overline{\psi}_{N,R}\psi_{N,L})-M(\psi_{N,R})^\tau\mathcal{C}\psi_{N,R} \\ & \qquad = \Big((\psi_{N,L})^\tau\mathcal{C}\quad(\psi_{N,R})^\tau\mathcal{C}\Big)\begin{pmatrix}0 & m\\ m & M\end{pmatrix}\begin{pmatrix}\psi_{N,L}\\ \psi_{N,R}\end{pmatrix}. \end{aligned} \tag{7.12}$$

The first contribution, with coefficient m, is the *Dirac mass* which in the Standard Model is induced by Yukawa coupling to the scalar doublet H, as for charged fermions: $m = \lambda_N v/\sqrt{2}$ in terms of the neutrino Yukawa coupling λ_N and of the vacuum expectation value v. Since there is no reason to expect that neutrino Yukawa couplings are much smaller than other Yukawa couplings, m should have similar magnitude as a quark or charged lepton mass. The second term is the *Majorana mass* which is invariant under G_{SM} only for the right-handed neutrino. The mass scale M is a free parameter of the Standard Model. The eigenvalues and eigenstates of the one-generation neutrino sector are then:

$$\begin{aligned} m_1 &= \frac{M}{2}\left(1-\sqrt{1+\frac{4m^2}{M^2}}\right), \qquad & \psi_1 &\sim \psi_{N,L}+\frac{M}{2m}\left(1-\sqrt{1+\frac{4m^2}{M^2}}\right)\psi_{N,R}, \\ m_2 &= \frac{M}{2}\left(1+\sqrt{1+\frac{4m^2}{M^2}}\right), \qquad & \psi_2 &\sim \frac{m}{M}\psi_{N,L}+\frac{1}{2}\left(1+\sqrt{1+\frac{4m^2}{M^2}}\right)\psi_{N,R}. \end{aligned} \tag{7.13}$$

Assume then $m \ll M$:

$$\begin{aligned} m_1 &\sim -\frac{m^2}{M}, \qquad & \psi_1 &\sim \psi_{N,L}-\frac{m}{M}\psi_{N,R} \\ m_2 &\sim M, \qquad & \psi_2 &\sim \psi_{N,R}+\frac{m}{M}\psi_{N,R}. \end{aligned} \tag{7.14}$$

We then find a very light neutrino state which is essentially the left-handed component which feels weak interactions $\psi_{N,L}$, and a heavy state, essentially made with $\psi_{N,R}$, with much weaker interactions[6] that other quarks and leptons (and then almost impossible to detect).

Returning to the three-generation case, the description of neutrino masses in the Standard Model uses the left-handed antineutrino spinors $\psi^{(n)}_{N^c,L}$ (or equivalently, the right-handed neutrinos $\psi^{(n)}_{N,R}$) and terms (5.19) of the Lagrangian. We have up to now defined neutrino fields ν_e, ν_μ and ν_τ from their weak couplings to charged leptons, using the unitary change of basis (7.8). We will work in this basis to obtain the neutrino mass matrix, which will then be non-diagonal in general. Mass terms following from (5.19) read then

$$\begin{aligned} -\frac{1}{2}\sum_{n,m=1}^{3}\Big(M^*_{nm}\,(\Psi^{(n)}_{N,R})^\tau C\Psi^{(m)}_{N,R}+\sqrt{2}v\tilde\lambda_{N\,nm}\overline{\Psi}^{(n)}_{N,R}\Psi^{(m)}_{N,L}\Big) \\ + \text{hermitian conjugate}\,, \end{aligned} \tag{7.15}$$

with $\tilde\lambda_{N\,nm} = \lambda_N^{np}(\mathcal{U}_E)^p_m$, noting however that the matrix $\mathcal{U}_E$ is not observable. The neutrino mass matrix involves then Majorana masses M_{mn} of the right-handed neu-

[6] There is a suppression factor $m/M \ll 1$.

trinos and the terms induced by the symmetry breaking and proportional to Yukawa couplings $\tilde{\lambda}_{N\,nm}$. It is the generalization of structure (7.12) to three generations and in the limit where the eigenvalues of M_{mn} are large with respect to $\tilde{\lambda}_{N,mn}v$, one will obtain three light, mostly left-handed, neutrino states (with weak interactions) and three heavy mostly right-handed states (without any interaction). Similarly to the CKM matrix for quarks, a mixing matrix will arise for leptons and, since the neutrino mass matrix involves even more free parameters than, for instance the charged lepton sector, masses and mixing are not predicted.

7.4 Parameters, numerical values

Without counting parameters describing neutrino masses and mixings, the Standard Model has eighteen (perturbative) arbitrary parameters:

- Three gauge coupling constants g_s, g and g'.
  Physical processes calculated in perturbation theory are actually functions of $g_s^2/4\pi$, $g^2/4\pi$ and ${g'}^2/4\pi$. The common practice is to characterize the three constants in terms of

  $$\alpha_s = \frac{g_s^2}{4\pi}, \qquad \text{(strong coupling constant)},$$

  $$\sin\theta_W = \frac{g'}{\sqrt{g^2+{g'}^2}}, \qquad (\theta_W : \text{Weinberg angle}),$$

  $$\alpha = \frac{e^2}{4\pi} \simeq (137)^{-1}, \qquad e = g\sin\theta_W, \qquad \text{(fine structure constant)}.$$

  Experimentally[7], $\sin^2\theta_W(M_Z) = .231$, $\alpha_s(M_Z) = .12$, $\alpha(m_e) = 137^{-1}$ (while $\alpha(M_Z) \simeq 128^{-1}$).

- The mass M_W of the weak boson $W^\pm$, or Fermi constant G_F.
  To lowest order in perturbation theory, these two quantities are related by.

  $$\frac{g^2}{8M_W^2} = \frac{1}{2v^2} = \frac{1}{\sqrt{2}}G_F.$$

  Weak interaction processes with (exchanged) energies much smaller than M_W provide a direct measure of G_F: $v = (\sqrt{2}G_F)^{-1/2} \simeq 246$ GeV and with $M_W = 80.4$ GeV, $g \simeq .65$ ($\frac{g^2}{4\pi} \simeq .034$).

[7] These coupling constants actually depend on energy.

- The nine masses of quarks u, d, s, c, b, t and leptons e, μ, τ.
  Since quarks only appear in colourless bound states (hadrons) measuring their masses is subtle and possibly ambiguous. Masses of light quarks u, d and s cannot be directly evaluated: these quarks form bound states where most of the mass, which is not well understood theoretically, is due to the nonperturbative dynamics of quantum chromodynamics, and not to the masses of the constituents of the bound states. The quark masses are then evaluated from their weak interactions: $m_u = 1.7 - 3.3$ MeV, $m_d = 4.1 - 5.8$ MeV and $m_s = 80 - 130$ MeV. Quarks c and b form more ordinary bound states. Masses of states $c\bar{c}$ (charmonium) and $b\bar{b}$ (bottomonium) in particular give a good evaluation of quark masses: $m_c = 1.18 - 1.34$ GeV and $m_b = 4.1 - 4.4$ GeV. The top quarks is so heavy that its bound states are too short-lived to provide useful information[8]. The value of its mass is then obtained from decay processes: $m_t = 172.0 \pm .9 \pm 1.3$ GeV.

- The four parameters (three angles and one CP violating phase) of the Cabibbo-Kobayashi-Maskawa matrix.
  Their values can be obtained from charged current weak interaction processes, mixing generations and CP violation is (hardly) accessible in systems[9] K^0–$\overline{K}^0$ and B^0–$\overline{B}^0$.

- The Higgs boson mass $m_h = \sqrt{\lambda}\, v$ [eq. (6.5)].
  The existence of the Higgs boson has not been established and its mass is then unknown. It is the first priority of experiments at LHC to either find the Higgs and measure its mass or disprove its existence within the energy range compatible with the Standard Model. Knowledge of m_h ialso gives the value of the couling constant λ which controls Higgs interactions.

[8] If they would be seen.

[9] Neutral mesons K^0 and B^0 are states $\bar{s}d$ and $\bar{b}d$.

Bibliography

[1] P. Ramond, "Group theory: a physicist's survey", Cambridge University Press, 2010.

[2] L. O'Raifeartaigh, "Group structure of gauge theories", Cambridge University Press, 1986.

[3] R. Gilmore, "Lie groups, physics and geometry: an introduction for physicists, engineers and chemists", Cambridge University Press, 2008.

[4] H. Georgi, "Lie algebras in particle physics: from isospin to unified theories",

[5] J. F. Cornwell, "Group theory in physics", 3 volumes, Academic Press, 1984 (I and II), 1989 (III).

[6] R. Gilmore, "Lie groups, Lie algebras and some of their applications", John Wiley and Sons, 1974; Dover Publications, 2006.

[7] B. C. Hall, "Lie groups, Lie algebras, and representations: an elementary introduction", Springer, 2003.

[8] W. Fulton and J. Harris, "Representation theory: a first course", Springer, 1991.

[9] J. E. Humphreys, "Introduction to Lie algebras and representation theory"; Springer, 1973.

[10] J. Goldstone, Nuovo Cim. **19** (1961) 154.

[11] Y. Nambu, Phys. Rev. Lett. **4** (1960) 380;
Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961);
J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. **127**, 965 (1962).

[12] F. Englert and R. Brout, Phys. Rev. Lett. **13** (1964) 321.

[13] P. W. Higgs, Phys. Rev. Lett. **13** (1964) 508.

[14] G. S. Guralnik, C. R. Hagen and T. W. B. Kibble, Phys. Rev. Lett. **13** (1964) 585.

[15] P. W. Higgs, Phys. Lett. **12** (1964) 132.

[16] S. L. Glashow, “Partial Symmetries Of Weak Interactions,” Nucl. Phys. **22** (1961) 579.

[17] A. Salam, “Weak and Electromagnetic Interactions”, in “Elementary Particle Theory”, ed. by N. Svartholm (Almqvist and Wiksell, Stockholm, 1968), p. 367.

[18] S. Weinberg, “A Model Of Leptons,” Phys. Rev. Lett. **19** (1967) 1264.

[19] J. C. Taylor, “Gauge theories of weak interactions,” *Cambridge University Press, 1976.*

[20] P. Renton, “Electroweak interactions: an introduction to the physics of quarks and leptons”, *Cambridge University Press, 1990.*

[21] F. Halzen and A. D. Martin, “Quarks and leptons: an introductory course in modern particle physics”, *John Wiley and Sons, 1984.*

[22] “Gauge theory of weak and electromagnetic interactions, selected papers”, edited by C. H. Lai, *World Scientific, Singapore, 1981.*